Math 228
4-15-1994

Solutions to Problem Set 31.

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2. Determine $\phi''(x_0)$, $\phi'''(x_0)$, and $\phi^{iv}(x_0)$ for the given point x_0 if $y = \phi(x)$ is a solution of the initial value problem

$$y'' + (\sin x)y' + (\cos x)y = 0, \qquad y(0) = 0, y'(0) = 1.$$

Plug in $x_0 = 0$:

$$y''(0) + (\sin 0)y'(0) + (\cos 0)y(0) = 0, \quad y''(0) = 0.$$

Differentiate the differential equation:

$$y''' + (\cos x)y' + (\sin x)y'' - (\sin x)y + (\cos x)y' = 0,$$

$$y''' + 2(\cos x)y' + (\sin x)y'' - (\sin x)y = 0.$$

Plug in $x_0 = 0$:

$$y'''(0) + 2(\cos 0)y'(0) + (\sin 0)y''(0) - (\sin 0)y(0) = 0, \quad y'''(0) = -2.$$

Differentiate the y''' equation:

$$y^{iv} - 2(\sin x)y' + 2(\cos x)y'' + (\cos x)y'' + (\sin x)y''' - (\cos x)y - (\sin x)y' = 0.$$

Plug in $x_0 = 0$:

$$y^{iv}(0) - 2(\sin 0)y'(0) + 2(\cos 0)y''(0) + (\cos 0)y''(0) + (\sin 0)y'''(0) - (\cos 0)y(0) - (\sin 0)y'(0) = 0, \quad y^{iv}(0) = 0. \square$$

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6. Determine a lower bound for the radius of convergence of series solutions about $x_0 = 4$, $x_0 = -4$, and $x_0 = 0$ for the differential equation

$$(x^2 - 2x - 3)y'' + xy' + 4y = 0.$$

Since $x^2 - 2x - 3 = (x-3)(x+1)$, there are singularities at $x = 3$ and at $x = -1$.
$x_0 = 4$ is 1 unit from $x = 3$, so the radius of convergence is $r = 1$.
$x_0 = -4$ is 3 units from $x = -1$, so the radius of convergence is $r = 3$.
$x_0 = 0$ is 1 unit from $x = -1$, so the radius of convergence is $r = 1$. $\square$

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7. Determine a lower bound for the radius of convergence of series solutions about $x_0 = 0$ and $x_0 = 2$ for the differential equation

$$(1 + x^3)y'' + 4xy' + y = 0.$$

$x^3 + 1 = (x+1)(x^2 - x + 1)$; the roots of $x^2 - x + 1$ are $\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$.

$x_0 = 0$ is 1 unit from $x = -1$ and 1 unit from $x = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$, so the radius of convergence is $r = 1$.

$x_0 = 2$ is $\sqrt{3}$ units from $x = \dfrac{1}{2} \pm \dfrac{\sqrt{3}}{2}i$, so the radius of convergence is $r = \sqrt{3}$. $\square$

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10. The Chebyshev differential equation is

$$(1-x^2)y'' - xy' + \alpha^2 y = 0,$$

where α is a constant.

(a) Determine two linearly independent solutions in power of x for $|x| < 1$.

Set

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Plug these into the differential equation:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=0}^{\infty} n(n-1)a_n x^n - \sum_{n=0}^{\infty} n a_n x^n + \alpha^2 \sum_{n=0}^{\infty} a_n x^n = 0.$$

In the second, third, and fourth sums, let $n = k-2$, so $n-1 = k-3$. When $n = 0$, $k = 2$, so the series become

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n = \sum_{k=2}^{\infty}(k-2)(k-3)a_{k-2}x^{k-2} = \sum_{n=2}^{\infty}(n-2)(n-3)a_{n-2}x^{n-2},$$

$$\sum_{n=0}^{\infty} n a_n x^n = \sum_{k=2}^{\infty}(k-2)a_{k-2}x^{k-2} = \sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2},$$

$$\alpha^2 \sum_{n=0}^{\infty} a_n x^n = \alpha^2 \sum_{k=2}^{\infty} a_{k-2}x^{k-2} = \alpha^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2}.$$

In each case, the second equality came from replacing the dummy variable k with n.
The differential equation is now

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} - \sum_{n=2}^{\infty}(n-2)(n-3)a_{n-2}x^{n-2} - \sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2} + \alpha^2 \sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0.$$

Since this is an identity in x, the coefficients of each power of x on the left side must be 0. For $n = 0$ and $n = 1$, this equation says $0 = 0$. For $n \geq 2$, I get

$$n(n-1)a_n - (n-2)(n-3)a_{n-2} - (n-2)a_{n-2} + \alpha^2 a_{n-2} = 0, \quad \text{or} \quad a_n = -\frac{(n-2)^2 - \alpha^2}{n(n-1)} a_{n-2}.$$

If I plug in even numbers for n, I get

$$\begin{aligned} a_2 &= -\frac{\alpha^2}{2} a_0, \\ a_4 &= \frac{4-\alpha^2}{4 \cdot 3} a_2, \\ &\vdots \\ a_{2n} &= \frac{(2n-2)^2 - \alpha^2}{2n(2n-1)} a_{2n-2}. \end{aligned}$$

Multiply these equations together — that is, multiply the left sides, multiply the right sides, and set the two products equal:

$$a_2 a_4 \cdots a_{2n} = \left(-\frac{\alpha^2}{2}\right)\left(\frac{4-\alpha^2}{4\cdot 3}\right)\cdots\left(\frac{(2n-2)^2-\alpha^2}{2n(2n-1)}\right) a_0 a_2 \cdots a_{2n-2}.$$

Cancel the terms a_2, a_4, …, a_{2n-2} from both sides:

$$a_{2n} = \left(-\frac{\alpha^2}{2}\right)\left(\frac{4-\alpha^2}{4\cdot 3}\right)\cdots\left(\frac{(2n-2)^2-\alpha^2}{2n(2n-1)}\right) a_0.$$

The product of the denominators of the fractions is $(2n)!$, so

$$a_{2n} = \frac{(-\alpha^2)(4-\alpha^2)\cdots((2n-2)^2-\alpha^2)}{(2n)!} a_0.$$

If I plug in odd numbers for n, I get

$$\begin{aligned} a_3 &= \frac{1-\alpha^2}{3\cdot 2} a_1, \\ a_5 &= \frac{9-\alpha^2}{5\cdot 4} a_3, \\ &\vdots \\ a_{2n+1} &= \frac{(2n-1)^2-\alpha^2}{(2n+1)(2n)} a_{2n-1}. \end{aligned}$$

Multiply the left sides, multiply the right sides, and set the two products equal:

$$a_3 a_5 \cdots a_{2n+1} = \left(\frac{1-\alpha^2}{3\cdot 2}\right)\left(\frac{9-\alpha^2}{5\cdot 4}\right)\cdots\left(\frac{(2n-1)^2-\alpha^2}{(2n+1)(2n)}\right) a_1 a_3 \cdots a_{2n-1}.$$

Cancel the terms a_3, a_5, …, a_{2n-1} from both sides:

$$a_{2n+1} = \left(\frac{1-\alpha^2}{3\cdot 2}\right)\left(\frac{9-\alpha^2}{5\cdot 4}\right)\cdots\left(\frac{(2n-1)^2-\alpha^2}{(2n+1)(2n)}\right) a_1.$$

The product of the denominators of the fractions is $(2n+1)!$, so

$$a_{2n+1} = \frac{(1-\alpha^2)(9-\alpha^2)\cdots((2n-1)^2-\alpha^2)}{(2n+1)!} a_1.$$

The solution is

$$y = a_0\left(1 + \sum_{n=1}^{\infty} \frac{(-\alpha^2)(4-\alpha^2)\cdots((2n-2)^2-\alpha^2)}{(2n)!} x^{2n}\right) +$$
$$a_1\left(x + \sum_{n=1}^{\infty} \frac{(1-\alpha^2)(9-\alpha^2)\cdots((2n-1)^2-\alpha^2)}{(2n+1)!} x^{2n+1}\right). \square$$

(b) Show that if α is a nonnegative integer n, then there is a polynomial solution of degree n. These polynomials, when properly normalized, are called the Chebyshev polynomials. They are very useful in problems requiring a polynomial approximation to a function defined on $-1 \le x \le 1$.

If $\alpha = 2m$, then $a_{2m+2} = 0$, so all the higher order even terms are 0. Then

$$y = 1 + \sum_{n=1}^{m} \frac{(-\alpha^2)(4-\alpha^2)\cdots((2n-2)^2-\alpha^2)}{(2n)!} x^{2n}$$

is a polynomial solution.

If $\alpha = 2m+1$, then $a_{2m+3} = 0$, so all the higher order odd terms are 0. Then

$$y = x + \sum_{n=1}^{m} \frac{(1-\alpha^2)(9-\alpha^2)\cdots((2n-1)^2-\alpha^2)}{(2n+1)!} x^{2n+1}$$

is a polynomial solution. □

(c) Find a polynomial solution for each of the cases $\alpha = n = 0, 1, 2$, and 3.

For $\alpha = 0$, $a_2 = 0$, and the higher order even terms are 0. $y = a_0$ is a solution.
For $\alpha = 1$, $a_3 = 0$, and the higher order odd terms are 0. $y = a_1 x$ is a solution.
For $\alpha = 2$, $a_4 = 0$, and the higher order even terms are 0. $y = a_0(1 - 2x^2)$ is a solution.
For $\alpha = 3$, $a_5 = 0$, and the higher order odd terms are 0. $y = a_1\left(x - \dfrac{4}{3}x^3\right)$ is a solution. □

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11. Find the first three terms in each of two linearly independent power series solutions in powers of x of

$$y'' + (\sin x)y = 0.$$

The Taylor series for $\sin x$ is

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots.$$

I will use the first three terms.
Set

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1} \quad \text{and} \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Plug these and the first three terms of the $\sin x$ series into the differential equation:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!}\right)\sum_{n=0}^{\infty} a_n x^n = 0,$$

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} a_n x^{n+1} - \frac{1}{3!}\sum_{n=0}^{\infty} a_n x^{n+3} + \frac{1}{5!}\sum_{n=0}^{\infty} a_n x^{n+5} = 0.$$

In the second series, let $n+1 = k-2$, so $n = k-3$. When $n = 0$, $k = 3$, and the series becomes

$$\sum_{n=0}^{\infty} a_n x^{n+1} = \sum_{k=3}^{\infty} a_{k-3}x^{k-2} = \sum_{n=3}^{\infty} a_{n-3}x^{n-2}.$$

The second equality follows by replacing the dummy variable k with n.

In the third series, let $n+3 = k-2$, so $n = k-5$. When $n = 0$, $k = 5$, and the series becomes

$$\frac{1}{3!}\sum_{n=0}^{\infty} a_n x^{n+3} = \frac{1}{3!}\sum_{k=5}^{\infty} a_{k-5} x^{k-2} = \frac{1}{3!}\sum_{n=5}^{\infty} a_{n-5} x^{n-2}.$$

The second equality follows by replacing the dummy variable k with n.

In the fourth series, let $n+5 = k-2$, so $n = k-7$. When $n = 0$, $k = 7$, and the series becomes

$$\frac{1}{5!}\sum_{n=0}^{\infty} a_n x^{n+5} = \frac{1}{5!}\sum_{k=7}^{\infty} a_{k-7} x^{k+2} = \frac{1}{5!}\sum_{n=7}^{\infty} a_{n-7} x^{n+2}.$$

The second equality follows by replacing the dummy variable k with n.

The differential equation now becomes

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=3}^{\infty} a_{n-3} x^{n-2} - \frac{1}{3!}\sum_{n=5}^{\infty} a_{n-5} x^{n-2} + \frac{1}{5!}\sum_{n=7}^{\infty} a_{n-7} x^{n+2} = 0.$$

Since this is an identity in x, the coefficients of each power of x on the left side must be 0. For $n = 0$ and $n = 1$, this equation says $0 = 0$. For $n = 2$, I get

$$2a_2 = 0, \quad \text{so} \quad a_2 = 0.$$

For $n = 3$, I get

$$6a_3 + a_0 = 0, \quad \text{so} \quad a_3 = -\frac{1}{6}a_0.$$

For $n = 4$, I get

$$12a_4 + a_1 = 0, \quad \text{so} \quad a_4 = -\frac{1}{12}a_1.$$

Continuing in this fashion, I find that

$$a_5 = \frac{1}{120}a_0 \quad , a_6 = \frac{1}{180}a_0 + \frac{1}{180}a_1, \quad a_7 = -\frac{1}{5040}a_0 + \frac{1}{504}a_1.$$

If I wanted a_9, I would need to go back and include the $\dfrac{x^7}{7!}$ term that I omitted from the $\sin x$ series.

The first few terms of the solution are therefore

$$y = a_0\left(1 - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{180}x^6 - \frac{1}{5040}x^7 + \cdots\right) + a_1\left(x - \frac{1}{12}x^4 + \frac{1}{180}x^6 + \frac{1}{504}x^7 + \cdots\right). \square$$

"Amusement is the happiness of those that cannot think." - Alexander Pope