Math 228
4-17-1994

Solutions to Problem Set 32.

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For each of the system in Problems 1 through 12:

(a) Find the eigenvalues and eigenvectors.

(b) Classify the critical point $(0,0)$ as to type and determine whether it is stable, asymptotically stable, or unstable.

(c) Sketch several trajectories in the phase plane and also sketch the graph of x_1 versus t.

(d) If possible, use a computer to plot accurately the curves requested in part (c).

I'll use *Mathematica*'s `Eigensystem` command to find the eigenvalues and eigenvectors. Instead of doing parts (c) and (d), I'll use *Mathematica* to sketch the direction field.

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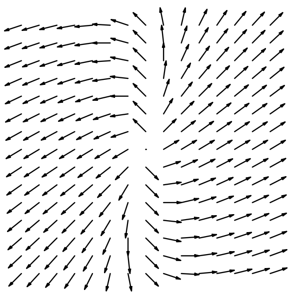
2. $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 5 & -1 \\ 3 & 1 \end{bmatrix} \vec{x}$

```
mat2 = {{5, -1}, {3, 1}}

    {{5, -1}, {3, 1}}

Eigensystem[mat2]

    {{2, 4}, {{1, 3}, {1, 1}}}
```



The origin is an unstable improper node. □

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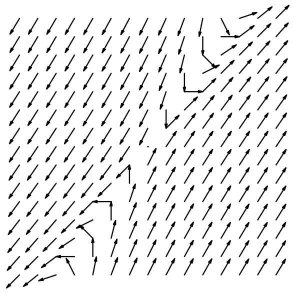
3. $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 2 & -1 \\ 3 & -2 \end{bmatrix} \vec{x}$

```
mat3 = {{2, -1}, {3, -2}}

    {{2, -1}, {3, -2}}

Eigensystem[mat3]
```

```
{{-1, 1}, {{1, 3}, {1, 1}}}
```



The origin is an unstable saddle. □

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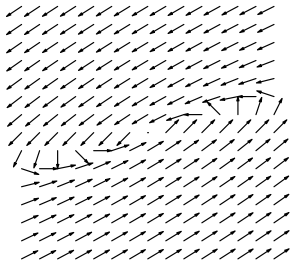
5. $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 1 & -5 \\ 1 & -3 \end{bmatrix} \vec{x}$

```
mat5 = {{1, -5}, {1, -3}}

    {{1, -5}, {1, -3}}

Eigensystem[mat5]

    {{-1 - I, -1 + I}, {{2 - I, 1}, {2 + I, 1}}}
```



The origin is an asymptotically stable spiral. □

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10. $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} 1 & 2 \\ -5 & -1 \end{bmatrix} \vec{x}$

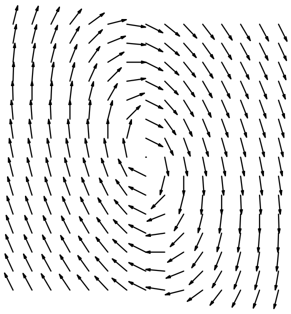
```
mat10 = {{1, 2}, {-5, -1}}

    {{1, 2}, {-5, -1}}

  Eigensystem[mat10]

                 1     3 I             1     3 I
```

```
{{ -3 I, 3 I}, {{ -(-) + ---, 1}, { -(-) - ---, 1}}}
                  5     5            5     5
```



The origin is a stable center. □

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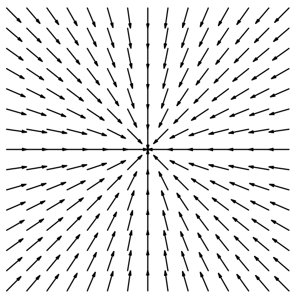
11. $\dfrac{d\vec{x}}{dt} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \vec{x}$

```
mat11 = {{-1, 0}, {0, -1}}

    {{-1, 0}, {0, -1}}

Eigensystem[mat11]

    {{-1, -1}, {{0, 1}, {1, 0}}}
```



The origin is an asymptotically stable proper node. □

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19. In this problem we indicate how to show that the trajectories are ellipses when the eigenvalues are pure imaginary. Consider the system

$$\begin{bmatrix} x \\ y \end{bmatrix}' = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \qquad (i)$$

(a) Show that the eigenvalues of the coefficient matrix are pure imaginary if and only if

$$a + d = 0, \quad ad - bc > 0. \qquad (ii)$$

The characteristic polynomial is $\lambda^2 - (a+d)\lambda + (ad - bc)$. The eigenvalues are
$$\lambda = \frac{(a+d) \pm \sqrt{(a+d)^2 - 4(ad-bc)}}{2}.$$
If the eigenvalues are pure imaginary, the real part must vanish: $a + d = 0$. This means that
$$\sqrt{(a+d)^2 - 4(ad-bc)} = \sqrt{-4(ad-bc)}.$$
Since this is an imaginary number, $ad - bc > 0$.
Conversely, suppose that $a + d = 0$ and $ad - bc > 0$. Then the eigenvalues are
$$\lambda = \frac{\pm\sqrt{-4(ad-bc)}}{2}.$$
Since $ad - bc > 0$, the eigenvalues are pure imaginary. □

(b) The trajectories of the system (i) can be found by converting Eqs. (i) into the single equation
$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{cx + dy}{ax + by}. \qquad (iii)$$
Use the first of Eqs. (ii) to show that Eq. (iii) is exact.

Equation (iii) can be written as
$$(cx + dy)\,dx - (ax + by)\,dy = 0.$$
Then $\dfrac{\partial M}{\partial y} = d$ and $\dfrac{\partial N}{\partial x} = -a$. But $a + d = 0$, so $a = -d$, and hence $\dfrac{\partial N}{\partial y} = \dfrac{\partial M}{\partial x}$. Therefore, the equation is exact. □

(c) By integrating Eq. (iii) show that
$$cx^2 + 2dxy - by^2 = k, \qquad (iv)$$
where k is a constant. Use Eqs. (ii) to conclude that the graph of Eq. (iv) is always an ellipse.

$$(cx + dy)\,dx - (ax + by)\,dy = 0.$$
Replace a with $-d$:
$$(cx + dy)\,dx - (-dx + by)\,dy = 0.$$
I'll find f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial F}{\partial y} = N$. First,
$$f = \int (cx + dy)\,dx = \frac{1}{2}cx^2 + dxy + g(y).$$
Differentiate with respect to y and set the result equal to N:
$$dx + \frac{dg}{dy} = dx - by, \quad \frac{dg}{dy} = -by.$$
It follows that $g = -\dfrac{1}{2}by^2$, so
$$f = \frac{1}{2}cx^2 + dxy - \frac{1}{2}by^2.$$
The solution is
$$\frac{1}{2}cx^2 + dxy - \frac{1}{2}by^2 = k_0, \quad \text{or} \quad cx^2 + 2dxy - by^2 = k.$$
Finally, $ad - bc = -d^2 - bc > 0$, so $d^2 + bc < 0$. Since d^2 is positive, this forces bc to be negative — that is, b and c have opposite signs. This means that the cx^2 and $-by^2$ terms in the equation end up with the same sign, so the equation represents an ellipse. □