Solutions to Problem Set 36.

[page 479]

Each of Problems 1 through 6 can be interpreted as describing the interaction of two species with populations x and y. In each of these problems carry out the following steps.

(a) Find the critical points.

(b) For each critical point find the corresponding linear system. Find the eigenvalues and eigenvectors of the linear system; classify each critical point as to type, and determine whether it is asymptotically stable, stable, or unstable.

(c) Sketch the trajectories in the neighborhood of each critical point.

(d) If possible, compute and plot enough trajectories of the given system to show clearly the behavior of the solutions.

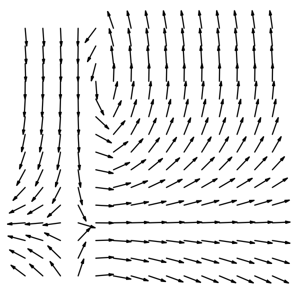
(e) Determine the limiting behavior of x and y as $t \to \infty$ and interpret the results in terms of the populations of the two species.

1.
$$\frac{dx}{dt} = x(1.5 - 0.5y)$$
$$\frac{dy}{dt} = y(-0.5 + x)$$

First, I'll find the critical points:

$$x(1.5 - 0.5y) = 0$$
$$\swarrow \qquad\qquad \searrow$$

$x = 0$	$y = 3$
$y(-0.5 + x) = 0$	$y(-0.5 + x) = 0$
$-0.5y = 0$	$3(-0.5 + x) = 0$
$y = 0$	$x = 0.5$
$(0, 0)$	$(0.5, 3)$



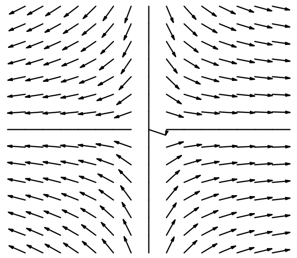
The matrix of partial derivatives is

$$A = \begin{bmatrix} 1.5 - 0.5y & -0.5x \\ y & -0.5 + x \end{bmatrix}.$$

At $(0,0)$,

$$A = \begin{bmatrix} 1.5 & 0 \\ 0 & -0.5 \end{bmatrix}.$$

$\det = -0.75 < 0$, so the critical point is a saddle, and it's unstable.

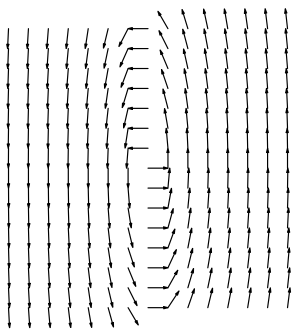


The eigenvalues are 1.5 and -0.5; the corresponding eigenvectors are $\langle 1, 0 \rangle$ and $\langle 0, 1 \rangle$.

At $(0.5, 3)$,

$$A = \begin{bmatrix} 0 & -0.25 \\ 3 & 0 \end{bmatrix}.$$

$\det = 0.75$ and $\mathrm{tr} = 0$; this case is indeterminate according to the method of almost linear systems. However, I know that this critical point of a predator-prey system is stable (e.g. by Liapunov stability).

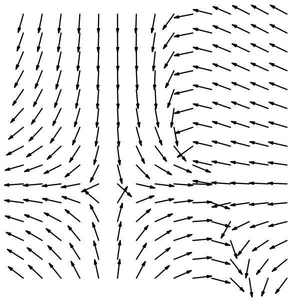


The eigenvalues are $\pm \frac{\sqrt{3}}{2} i$; the corresponding eigenvectors are $\langle 1, \pm 2i\sqrt{3} \rangle$. □

4.
$$\frac{dx}{dt} = x(1.125 - x - 0.5y)$$
$$\frac{dy}{dt} = y(-1 + x)$$

First, I'll find the critical points:

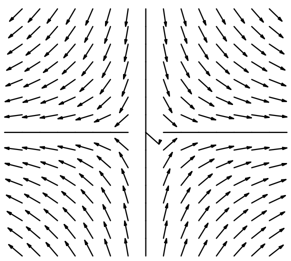
$$\begin{array}{ccc}
 & y(-1+x)=0 & \\
\swarrow & & \searrow \\
y=0 & & x=1 \\
x(1.125-x-0.5y)=0 & & x(1.125-x-0.5y)=0 \\
x(1.125-x)=0 & & 0.125-0.5y=0 \\
\downarrow & \searrow & y=0.25 \\
x=0 & x=1.125 & (1,0.25) \\
(0,0) & (1,125,0) &
\end{array}$$



At $(0,0)$,

$$A = \begin{bmatrix} 1.125 & 0 \\ 0 & -1 \end{bmatrix}.$$

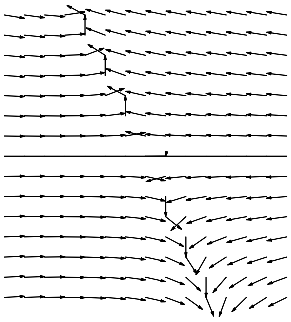
$\det = -1.125 < 0$, so the critical point is a saddle, and it's unstable.



The eigenvalues are 1.125 and -1; the corresponding eigenvectors are $\langle 1,0\rangle$ and $\langle 0,1\rangle$.
At $(1.125,0)$,

$$A = \begin{bmatrix} -1.125 & -0.5625 \\ 0 & 0.125 \end{bmatrix}.$$

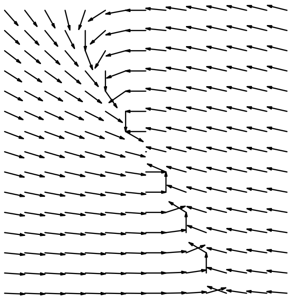
$\det = -\dfrac{9}{64} < 0$, so the critical point is a saddle, and it's unstable.



The eigenvalues are -1.125 and 0.125; the corresponding eigenvectors are $\langle 1, 0\rangle$ and $\langle 1, 2\rangle$.
At $(1, 0.25)$,

$$A = \begin{bmatrix} -1 & -0.5 \\ 0.25 & 0 \end{bmatrix}.$$

$\det = 0.125$ and $\operatorname{tr} = -1$, so $\det < \dfrac{1}{4}\operatorname{tr}^2$. The critical point is an asymptotically stable node.



The eigenvalues are $-\dfrac{1}{2} \pm \dfrac{\sqrt{2}}{2}$; the corresponding eigenvectors are $\left\langle 2 \mp 2\sqrt{2}, 1\right\rangle$. $\square$

"Self-sacrifice enable us to sacrifice other people without blushing." - George Bernard Shaw