Solutions to Problem Set 3.

[page 28]

5. Find the solution of the initial value problem $xy' + 2y = x^2 - x + 1$, $y(1) = \frac{1}{2}$. State the integral in which the solution is valid.

Write the equation in the form $y' + \frac{2}{x}y = x - 1 + \frac{1}{x}$. The integrating factor is

$$I = \exp \int \frac{2}{x}\,dx = \exp 2\ln x = \exp \ln x^2 = x^2.$$

Therefore,

$$yx^2 = \int (x^3 - x^2 + x)\,dx = \frac{1}{4}x^4 - \frac{1}{3}x^3 + \frac{1}{2}x^2 + C.$$

The solution is

$$y = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{C}{x^2}.$$

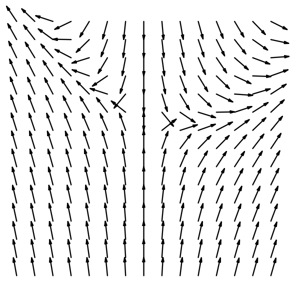
Plug in the initial condition:

$$\frac{1}{2} = y(1) = \frac{1}{4} - \frac{1}{3} + \frac{1}{2} + C \quad \text{so} \quad C = \frac{1}{12}.$$

The solution is

$$y = \frac{1}{4}x^2 - \frac{1}{3}x + \frac{1}{2} + \frac{1}{12}\frac{1}{x^2}.$$

The picture below shows the direction field for the equation. Notice the discontinuity at $x = 0$.



$P(x) = \frac{2}{x}$, $Q(x) = x - 1 + \frac{1}{x}$ are continuous for $x \neq 0$. The initial condition is $x = 1$, $y = \frac{1}{2}$. Since $x = 1 > 0$, so the solution is valid for $x > 0$. □

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9. Find the solution of the initial value problem $y' + (\cot x)y = 4\sin x$, $y\left(-\frac{\pi}{2}\right) = 0$. State the integral in which the solution is valid.

The integrating factor is

$$I = \exp \int \cot x\,dx = \exp \int \frac{\cos x}{\sin x}\,dx = \exp \ln \sin x = \sin x.$$

Hence,

$$y \sin x = \int 4(\sin x)^2 \, dx = 2 \int (1 - \cos 2x) \, dx = 2(x - \frac{1}{2} \sin 2x) + C = 2(x - \sin x \cos x) + C.$$

The solution is

$$y = \frac{2x}{\sin x} - 2 \cos x + \frac{C}{\sin x}.$$

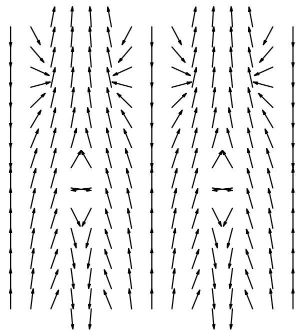
Plug in the initial condition:

$$0 = y\left(-\frac{\pi}{2}\right) = \frac{-\pi}{-1} - 0 + \frac{C}{-1} \quad \text{so} \quad C = \pi.$$

The solution is

$$y = \frac{2x}{\sin x} - 2 \cos x + \frac{\pi}{\sin x}.$$

The picture below shows the direction field for the equation. The periodic vertical "lines" reflect the fact that $\cot x$ blows up when x is a multiple of π.



$P(x) = \cot x$, $Q(x) = 4 \sin x$ are continuous for $x \neq n\pi$, where n is an integer. The interval containing $x = -\frac{\pi}{2}$ is $-\pi < x < 0$. Hence, the solution is valid on the interval$-\pi < x < 0$. □

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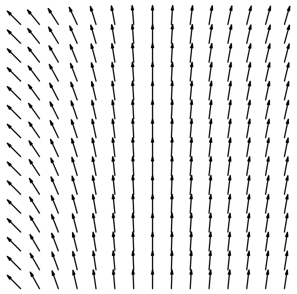
17. Determine (without solving the problem) an interval in which the solution of the initial value problem $(x - 3)y' + (\ln x)y = 2x$, $y(1) = 2$ is certain to exist.

Rewrite the equation as $y' + \frac{\ln x}{x - 3} y = \frac{2x}{x - 3}$. $P = \frac{\ln x}{x - 3}$ is defined for $0 < x < 3$ and for $x > 3$; $Q = \frac{2x}{x - 3}$ is defined for $x > 3$ and for $x < 3$. They are both continuous on the intervals $0 < x < 3$ and $x > 3$. The initial condition is $x = 1$, $y = 2$. Since $x = 1$ lies in $0 < x < 3$, the largest interval on which the existence and uniqueness theorem guaranteed a solution to the initial value problem is $0 < x < 3$.

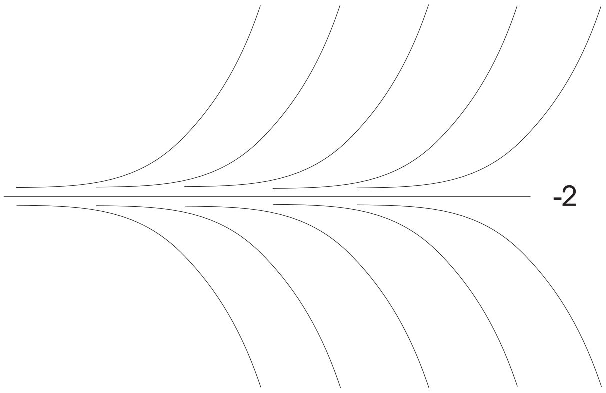
Here is the direction field for the equation. Notice the discontinuities at $x = 0$ and at $x = 3$.



□

21. For the initial value problem $y' - y = 2$, $y(0) = y_0$, determine how the limiting value of y as $x \to \infty$ depends on y_0.

Since $y' = y + 2$, the solution curves increase for $y > -2$ and decrease for $y < -2$. Hence, the phase portrait looks like this:



The limiting behavior of the solutions is evident from the picture. If $y_0 > -2$ — that is, if the solution curve crosses the y-axis above $y = -2$ — then $y \to +\infty$ as $x \to \infty$. If $y_0 < -2$, then $y \to -\infty$ as $x \to \infty$. If $y_0 = -2$, then $y = -2$, so $y \to -2$ as $x \to \infty$. □

25. Show that if a and λ are positive contants, and b is any real number, then every solution of the equation

$$y' + ay = be^{-\lambda x}$$

has the property that $y \to 0$ as $x \to \infty$.

The integrating factor is

$$I = \exp \int x \, dx = e^{ax}.$$

Hence,

$$ye^{ax} = b \int e^{(a-\lambda)x} \, dx.$$

There are two cases to consider.
Suppose $a = \lambda$. The equation becomes

$$ye^{ax} = b \int dx = bx + C.$$

The solution is

$$y = bxe^{-ax} + Ce^{-ax}, \quad \text{and} \quad \lim_{x \to \infty} (bxe^{-ax} + Ce^{-ax}) = 0.$$

Suppose that $a \neq \lambda$. The equation becomes

$$ye^{ax} = b \int e^{(a-\lambda)x} \, dx = \frac{b}{a - \lambda} e^{(a-\lambda)x} + C.$$

The solution is

$$y = \frac{b}{a - \lambda} e^{-\lambda x} + Ce^{-ax}, \quad \text{and} \quad \lim_{x \to \infty} (\frac{b}{a - \lambda} e^{-\lambda x} + Ce^{-ax}) = 0. \; □$$

26. **Discontinuous coefficients.** Linear differential equations sometimes occur in which one or both of the function p and q have jump discontinuities. If x_0 is such a point of discontinuity, then it is necessary to solve the equation separately for $x < x_0$ and for $x > x_0$. Afterward the two solutions are matched so that y is continuous at x_0; this is accomplished by a proper choice of the arbitrary constants. The following two problems illustrate this situation. Notice in each case that it is impossible also to make y' continuous at x_0.

(a) Solve the initial value problem

$$y' + 2y = g(x), \quad y(0) = 0, \quad \text{where} \quad g(x) = \begin{cases} 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

The integrating factor is

$$I = \exp \int 2\, dx = e^{2x}.$$

Hence,

$$ye^{2x} = \int g(x) e^{2x}\, dx.$$

There are two cases. First, suppose $0 \le x \le 1$. $g = 1$, so

$$ye^{2x} = \int e^{2x}\, dx = \frac{1}{2} e^{2x} + C.$$

The solution is

$$y = \frac{1}{2} + Ce^{-2x}.$$

Plug in the initial condition. (Note that $x = 0$ is in this interval!)

$$0 = y(0) = \frac{1}{2} + C \quad \text{so} \quad C = -\frac{1}{2}.$$

The solution on this interval is

$$y = \frac{1}{2} - \frac{1}{2} e^{-2x}.$$

Note for reference that $y(1) = \frac{1}{2} - \frac{1}{2} e^{-2}$.

Now suppose that $x > 1$. $g = 0$, so

$$ye^{2x} = \int 0\, dx = C.$$

The solution is $y = Ce^{-2x}$.

In order for the solution to be continuous at $x = 1$, it must agree with the left half at $x = 1$. This means that $y(1) = \frac{1}{2} - \frac{1}{2} e^{-2}$. This ensures that the left end of this piece matches the right end of the previous piece. Insert this initial condition:

$$\frac{1}{2} - \frac{1}{2} e^{-2} = Ce^{-2} \quad \text{so} \quad C = \frac{1}{2} e^{2} - \frac{1}{2}.$$

The solution for $x > 1$ is

$$y = \left(\frac{1}{2} e^{2} - \frac{1}{2} \right) e^{-2x}.$$

The whole solution is obtained by pasting the two pieces together:

$$y = \begin{cases} \dfrac{1}{2} - \dfrac{1}{2} e^{-2x} & 0 \le x \le 1 \\ \left(\dfrac{1}{2} e^{2} - \dfrac{1}{2} \right) e^{-2x} & x > 1 \end{cases}$$

(Note, however, that y' is *not* continuous at $x = 1$.) $\square$

(b) Solve the initial value problem

$$y' + p(x)y = 0, \quad y(0) = 1, \quad \text{where} \quad p(x) = \begin{cases} 2 & 0 \le x \le 1 \\ 1 & x > 1 \end{cases}$$

For $0 \le x \le 1$, the equation is $y' + 2y = 0$, or $y' = -2y$. This is the standard exponential decay equation, so the solution is $y = Ce^{-2x}$. Since $y(0) = 1$, $C = 1$, and $y = e^{-2x}$. This is the solution on the interval $0 \le x \le 1$.

Note that $y(1) = e^{-2}$.

Now suppose $x > 1$. The equation is $y' + y = 0$, so $y' = -y$. This time, we obtain $y = Ce^{-x}$. Using the condition $y(1) = e^{-2}$ from the first piece, it follows that $C = e^{-1}$. The solution is $y = e^{-1}e^{-x} = e^{-(x+1)}$.

Paste the two pieces together to obtain the whole solution:

$$y = \begin{cases} e^{-2x} & \text{if } 0 \le x \le 1 \\ e^{-(x+1)} & \text{if } x > 1 \end{cases} \quad \square$$

"It is often the failure who is the pioneer in new lands, new undertakings, and new forms of expression."
Eric Hoffer

