Math 228
1-26-1994

Solutions to Problem Set 4.

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3. Solve the differential equation $y' + y^2 \sin x = 0$.

Separate and integrate:

$$\int \frac{1}{y^2}\,dy = -\int \sin x\,dx, \quad -\frac{1}{y} = \cos x + C, \quad y = -\frac{1}{\cos x + C}. \ \Box$$

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4. Solve the differential equation $y' = 1 + x + y^2 + xy^2$.

Separate and integrate:

$$\int \frac{dy}{1+y^2} = \int (1+x)\,dx, \quad \arctan y = x + \frac{1}{2}x^2 + C, \quad y = \tan\left(x + \frac{1}{2}x^2 + C\right). \ \Box$$

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5. Solve the differential equation $y' = (\cos x)^2(\cos 2y)^2$.

Separate and integrate:

$$\int (\sec 2y)^2\,dy = \int (\cos x)^2\,dx = \frac{1}{2}\int (1 + \cos 2x)\,dx, \quad \frac{1}{2}\tan 2y = \frac{x}{2} + \frac{1}{4}\sin 2x + C. \ \Box$$

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13. Find the solution of the initial value problem $y' = \dfrac{2x}{1+2y}$, $y(2) = 0$, in explicit form, and determine (at least approximately) the interval in which it is defined.

Separate and integrate:

$$\int (1+2y)\,dy = \int 2x\,dx, \quad y + y^2 = x^2 + C.$$

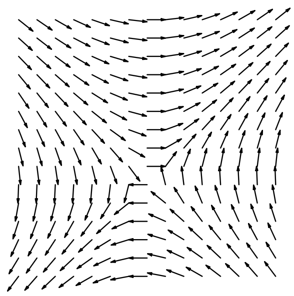
Plug in the initial condition:

$$0 + 0 = 4 + C, \quad C = -4.$$

The solution is

$$y + y^2 = x^2 - 4.$$

The picture below shows the direction field for the equation:



To obtain an "explicit" solution, write $y^2 + y - (x^2 - 4) = 0$. By the general quadratic formula,

$$y = \frac{-1 \pm \sqrt{1 + 4(x^2 - 4)}}{2}.$$

$y(2) = 0$ requires the plus sign. Therefore,

$$y = \frac{-1 + \sqrt{4x^2 - 15}}{2}.$$

Now

$$f(x, y) = \frac{2x}{1 + 2y}, \quad \frac{\partial f}{\partial y} = \frac{-4x}{(1 + 2y)^2}.$$

f and $\dfrac{\partial f}{\partial y}$ are continuous for $y \neq -\dfrac{1}{2}$. The initial condition is $y(2) = 0$, the interval is $y > -\dfrac{1}{2}$. What is the corresponding range for x?

$$-\frac{1}{2} < \frac{-1 + \sqrt{4x^2 - 15}}{2}, \quad 4x^2 > 15, \quad x < -\frac{\sqrt{15}}{2} \quad \text{or} \quad x > \frac{\sqrt{15}}{2}.$$

The initial condition had $x = 2$, which is in the second of these intervals. Hence, the solution is valid for $x > \dfrac{\sqrt{15}}{2}$. □

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17. Solve the initial value problem

$$y' = \frac{1 + 3x^2}{3y^2 - 6y}, \quad y(0) = 1$$

and determine the interval in which the solution is valid.

Separate and integrate:

$$\int (3y^2 - 6y)\, dy = \int (1 + 3x^2)\, dx, \quad y^3 - 3y^2 = x + x^3 + C.$$

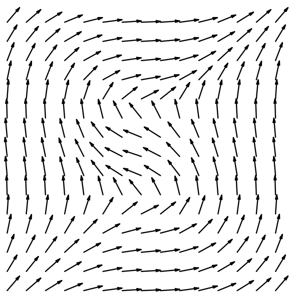
Plug in the initial condition:

$$1 - 3 = 0 + 0 + C, \quad C = -2.$$

The solution is

$$y^3 - 3y^2 = x + x^3 - 2.$$

The picture below shows the direction field for the equation:



Now

$$f(x,y) = \frac{1+3x^2}{3y^2-6y}, \quad \frac{\partial f}{\partial y} = \frac{-(6y-6)(1+3x^2)}{(3y^2-6y)^2}.$$

Since $3y^2 - 6y = 3y(y-2)$, f and $\frac{\partial f}{\partial y}$ are continuous on $y < 0$, $0 < y < 2$, and $y > 2$. $y(0) = 1$ is in the interval $0 < y < 2$. What is the range for x?

Setting $y = 0$ in $y^3 - 3y^2 = x + x^3 - 2$ gives $x = 1$. (The other roots of the cubic are imaginary.)

Setting $y = 2$ in $y^3 - 3y^2 = x + x^3 - 2$ gives $x = -1$. (The other roots of the cubic are imaginary.)

These values are the endpoints of the intervals of validity of the solutions. Note that $x = 0$, the initial condition value, is in the interval $-1 < x < 1$. This is the interval on which the solution is valid. □

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2. State the region in the x-y-plane where the hypotheses of Theorem 2.4.1 are satisfied for the equation $y' = (1 - x^2 - y^2)^{1/2}$. Thus there is a unique solution through each given initial point in this region.

There is a unique solution to the equation $y' = f(x,y)$ which passes through a point (x_0, y_0) provided that (x_0, y_0) lies in an open rectangle where f and $\frac{\partial f}{\partial y}$ are continuous.

In this problem,

$$f(x,y) = \sqrt{1 - x^2 - y^2}, \quad \frac{\partial f}{\partial y} = -\frac{y}{\sqrt{1 - x^2 - y^2}}.$$

f and $\frac{\partial f}{\partial y}$ are continuous for $x^2 + y^2 < 1$, the interior of the unit disk. Therefore, for any point in $x^2 + y^2 < 1$, there is a unique solution passing through the point. □

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4. State the region is the x-y-plane where the hypotheses of Theorem 2.4.1 are satisfied for the equation $y' = \frac{3}{(x+y)^2}$. Thus there is a unique solution through each given initial point in this region.

There is a unique solution to the equation $y' = f(x,y)$ which passes through a point (x_0, y_0) provided that (x_0, y_0) lies in an open rectangle where f and $\frac{\partial f}{\partial y}$ are continuous.

In this problem,

$$f(x,y) = \frac{3}{(x+y)^2}, \quad \frac{\partial f}{\partial y} = -\frac{6}{(x+y)^3}.$$

f and $\frac{\partial f}{\partial y}$ are continuous for $x + y \neq 0$. Therefore, given any point (x, y) for which $x + y \neq 0$, there is a unique solution passing through the point. □

"Consistency requires you to be as ignorant today as you were a year ago." - Bernard Berenson