Solutions to Problem Set 5.

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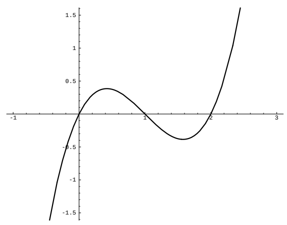
3. Sketch $\frac{dN}{dt}$ versus N for the equation

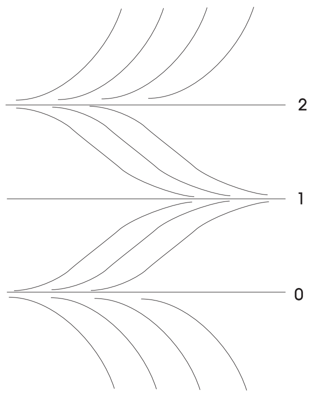
$$\frac{dN}{dt} = N(N-1)(N-2), \quad N_0 \geq 0;$$

determine the critical (equilibrium) points and classify each one as stable or unstable.

This is the graph of $\frac{dN}{dt}$ versus N:



From this, it is easy to draw the phase portrait. The curves must increase for $N > 2$ and for $0 < N < 1$. The curves decrease for $N < 0$ and for $1 < N < 2$.



$N = 0$ and $N = 2$ are unstable equilibria; $N = 1$ is stable. □

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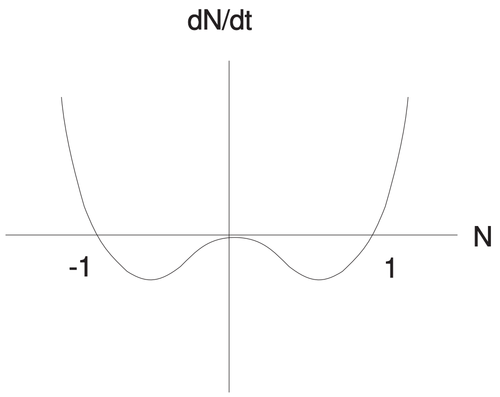
9. Sketch $\frac{dN}{dt}$ versus N for the equation

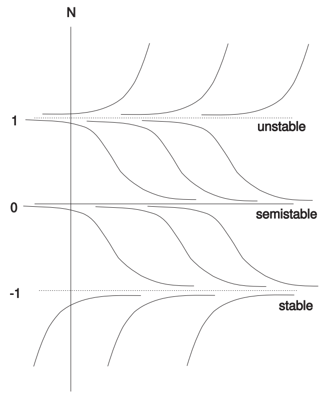
$$\frac{dN}{dt} = N^2(N^2-1), \quad -\infty < N_0 < +\infty;$$

determine the critical (equilibrium) points and classify each one as stable, unstable, or semistable.

The graph of $\frac{dN}{dt}$ against N is shown below:



It's clear from the graph that N increases for $N > 1$ and $N < -1$, and N decreases for $-1 < N < 1$.



$N = 1$ is an unstable solution, $N = 0$ is a semistable solution, and $N = -1$ is a stable solution. $\square$

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15. Suppose that a certain population of bunny rabbits obeys the logistic equation $\frac{dN}{dt} = rN\left[1 - \frac{N}{K}\right]$.

(a) If $N_0 = \frac{K}{3}$, find the time τ at which the initial population has doubled. Find the value of τ corresponding to $r = 0.025$ per year.

The general solution to the logistic equation is

$$N = \frac{K}{C_1 e^{-rt} + 1}.$$

When $t = 0$, $N = \frac{K}{3}$:

$$\frac{K}{3} = \frac{K}{C_1 + 1}, \quad C_1 = 2.$$

The equation is

$$N = \frac{K}{2e^{-rt} + 1}.$$

The question asks when the initial population has doubled, i.e. $N = \dfrac{2K}{3}$:

$$\frac{2K}{3} = \frac{K}{2e^{-rt}+1}, \quad t = \frac{\ln 4}{r}.$$

If $r = 0.025$, then $t = 40 \ln 4 \approx 55.45177$. □

(b) If $\dfrac{N_0}{K} = \alpha$, find the time T at which $\dfrac{N(T)}{K} = \beta$, where $0 < \alpha, \beta < 1$. Observe that $T \to \infty$ as $\alpha \to 0$ or as $\beta \to 1$. Find the value of T for $r = 0.025$ per year, $\alpha = 0.1$, and $\beta = 0.9$.

When $t = 0$, $N = \alpha K$:

$$\alpha K = \frac{K}{C_1 + 1}, \quad C_1 = \frac{1}{\alpha} - 1.$$

The equation is

$$N = \frac{K}{\left(\dfrac{1}{\alpha} - 1\right) e^{-rt} + 1}.$$

Set $\dfrac{N(T)}{K} = \beta$:

$$\frac{N(T)}{K} = \frac{1}{\left(\dfrac{1}{\alpha} - 1\right) e^{-rt} + 1}, \quad t = \frac{1}{r} \ln \frac{\beta - \alpha\beta}{\alpha - \alpha\beta}.$$

Set $r = 0.025$, $\alpha = 0.1$, and $\beta = 0.9$: $t \approx 175.77797$. □

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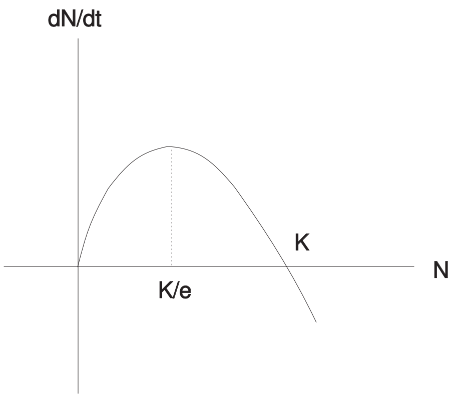
16. Another equation that has been used to model population growth is the Gompertz equation:

$$\frac{dN}{dt} = rN \ln \frac{K}{N},$$

where r and K are positive constants.

(a) Sketch the graph of $\dfrac{dN}{dt}$ versus N, find the critical points, and determine whether each is stable or unstable.

This is the graph of $\dfrac{dN}{dt}$ versus N:

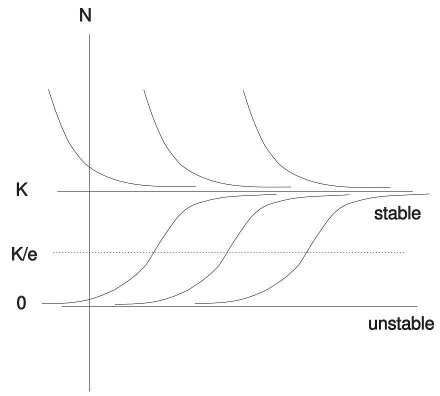


Note that by L'Hôpital's rule,

$$\lim_{N\to 0+} N\ln\frac{K}{N} = \lim_{N\to 0+}\frac{\ln\frac{K}{N}}{\frac{1}{N}} = \lim_{N\to 0+}\frac{\frac{N}{K}\cdot\frac{-K}{N^2}}{-\frac{1}{N^2}} = \lim_{N\to 0+} N = 0.$$

Hence, while $\frac{dN}{dt}$ isn't defined for $N = 0$, it approaches 0 as $N \to 0+$.

It's clear from the graph that N increases for $0 < N < K$, and N decreases for $K < N$. There will be inflection points for $N = \frac{K}{e}$.



$N = K$ is a stable solution, while $N = 0$ is an unstable solution. □

(b) For $0 \le N \le K$ determine where the graph of N versus t is concave up and where it is concave down.

$$\frac{d^2N}{dt^2} = r\left(-1+\ln\frac{K}{N}\right)\frac{dN}{dt} = r^2\left(-1+\ln\frac{K}{N}\right)\cdot N\ln\frac{K}{N}.$$

Therefore, the curves are concave up for $N > K$ and $0 < N < \frac{K}{e}$, and concave down for $\frac{K}{e} < N < K$. □

(c) For each N in $0 < N \le K$ show that $\frac{dN}{dt}$ as given by the Gompertz equation is never less than $\frac{dN}{dt}$ as given by the logistic equation.

I want to show that for $0 < N \le K$,

$$\left(\frac{dN}{dt}\right)_{\text{Gompertz}} \ge \left(\frac{dN}{dt}\right)_{\text{logistic}}.$$

That is, populations growing according to the Gompertz model grow faster than populations obeying the logistic model, and long as the populations are less than the limiting population.

I need to show that $\ln\frac{K}{N} \ge 1 - \frac{N}{K}$ for $0 < N \le K$. Write $f(N) = \ln\frac{K}{N} - 1 + \frac{N}{K}$. Then

$$f'(N) = -\frac{1}{N} + \frac{1}{K} < 0, \quad \text{and} \quad f(K) = 0.$$

Since f decreases on $[0, K]$ and equals 0 at the right-hand endpoint $N = K$, it follows that $f \ge 0$ for $0 < N \le K$. Therefore,

$$\ln\frac{K}{N} - 1 + \frac{N}{K} \ge 0, \quad \ln\frac{K}{N} \ge 1 - \frac{N}{K}, \quad rN\ln\frac{K}{N} \ge rN\left(1 - \frac{N}{K}\right).$$

This shows that
$$\left(\frac{dN}{dt}\right)_{\text{Gompertz}} \ge \left(\frac{dN}{dt}\right)_{\text{logistic}}. \quad \square$$

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17. (a) Solve the Gompertz equation
$$\frac{dN}{dt} = rN\ln\frac{K}{N},$$
subject to the initial condition $N(0) = N_0$.

Separate variables:
$$\frac{dN}{dt} = rN(\ln K - \ln N), \quad \int \frac{1}{N(\ln K - \ln N)}\,dN = \int r\,dt, \quad -\ln|\ln K - \ln N| = rt + C.$$

Isolate $\ln N$:
$$\ln|\ln K - \ln N| = -rt - C, \quad \ln K - \ln N = C_0e^{-rt}, \quad \ln N = \ln K - C_0e^{-rt}.$$

The solution is
$$N = \exp\left(\ln K - C_0e^{-rt}\right) = K\exp\left(C_1e^{-rt}\right).$$

Plug in the initial condition:
$$N_0 = N(0) = K\exp(tC_1), \quad C_1 = \ln\frac{N_0}{K}.$$

The solution is
$$N = K\exp\left(\left(\ln\frac{N_0}{K}\right)e^{-rt}\right). \quad \square$$

(b) For the data given in Example 1 in the text [$r = 0.71$ per year, $K = 80.5 \times 10^6$ kg, $\dfrac{N_0}{K} = 0.25$], use the Gompertz model to find the predicted value of $N(2)$.

Set $r = 0.71$, $K = 80.5 \times 10^6$, and $\dfrac{N_0}{K} = 0.25$:
$$N(2) = (80.5 \times 10^6)\exp\left((\ln 0.25)\,e^{-(0.71)(2)}\right) \approx 57.57972 \times 10^6. \quad \square$$

(c) For the same data as in part (b), use the Gompertz model to find the time τ at which $N(\tau) = 0.75K$.

$$0.75K = K\exp\left((\ln 0.25)\,e^{-0.71t}\right), \quad \ln 0.75 = (\ln 0.25)\,e^{-0.71t}, \quad e^{-0.71t} = \frac{\ln 0.75}{\ln 0.25} \quad -0.71t = \ln\frac{\ln 0.75}{\ln 0.25}.$$

Therefore, $t = -\dfrac{1}{0.71}\ln\dfrac{\ln 0.75}{\ln 0.25} \approx 2.21484.$ $\square$

"To do a great work a man must be very idle as well as very industrious." - Samuel Butler (II)