Math 228
1-29-1994

Solutions to Problem Set 6.

[page 80]

3. Determine whether or not the equation $(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$ is exact. If it is exact, find the solution.

$$M_y = -2x \quad \text{and} \quad N_x = -2x$$

so the equation is exact.

I need to find f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$. Integrate the first equation with respect to x:

$$f = \int (3x^2 - 2xy + 2)\,dx = x^3 - x^2y + 2x + h(y).$$

Differentiate f with respect to y and set the result equal to N:

$$6y^2 - x^2 + 3 = \frac{\partial f}{\partial y} = -x^2 + \frac{dh}{dy}.$$

Therefore,

$$\frac{dh}{dy} = 6y^2 + 3 \quad \text{so} \quad h = 2y^3 + 3y.$$

Hence, $f = x^3 - x^2y + 2x + 2y^3 + 3y$. The solution is

$$x^3 - x^2y + 2x + 2y^3 + 3y = C. \ \square$$

Remark. Note that in integrating to find h you don't need to add an arbitrary constant, since one will be inserted in the final step. However, *don't omit the arbitrary constant in the last line!* — the answer is incomplete without it. In addition, you'll get very confused the first time you have an initial value problem. $\square$

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8. Determine whether or not the equation $(e^x \sin y + 3y)\,dx - (3x - e^x \sin y)\,dy = 0$ is exact. If it is exact, find the solution.

$$M_y = e^x \cos y + 3 \quad \text{and} \quad N_x = -3 + e^x \sin y.$$

The equation is not exact. $\square$

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11. Determine whether or not the equation $(x \ln y + xy)\,dx + (y \ln x + xy)\,dy = 0$, $x > 0$, $y > 0$, is exact. If it is exact, find the solution.

$$M_y = \frac{x}{y} + x \quad \text{and} \quad N_x = \frac{y}{x} + y.$$

The equation is not exact. $\square$

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13. Solve the initial value problem $(2x-y)\,dx+(2y-x)\,dy = 0$, $y(1) = 3$, and determine at least approximately where the solution is valid.

$$M_y = -1 \quad \text{and} \quad N_x = -1$$

so the equation is exact.

I need to find f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$. Integrate the first equation with respect to x:

$$f = \int (2x - y)\,dx = x^2 - xy + h(y).$$

Differentiate f with respect to y and set the result equal to N:

$$2y - x = \frac{\partial f}{\partial y} = -x + \frac{dh}{dy}.$$

Therefore,

$$\frac{dh}{dy} = 2y \quad \text{so} \quad h = y^2$$

Hence, $f = x^2 - xy + y^2$. The solution is

$$x^2 - xy + y^2 = C.$$

Plug in the initial condition:

$$1 - 3 + 9 = C, \quad C = 7.$$

The solution is

$$x^2 - xy + y^2 = 7.$$

Solve for y: $y = \dfrac{x \pm \sqrt{28 - 3x^2}}{2}$. $y(1) = 3$ is satisfied for the positive square root, so $y = \dfrac{x + \sqrt{28 - 3x^2}}{2}$.

This solution is defined on the open interval where $28 - 3x^2 > 0$, i.e. for $-\sqrt{\dfrac{28}{3}} < x < \sqrt{\dfrac{28}{3}}$. $\square$

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15. Find the value of b for which the equation $(xy^2 + bx^2y)\,dx + (x + y)x^2\,dy = 0$ is exact, and then solve it using that value of b.

$$M_y = 2xy + bx^2 \quad \text{and} \quad N_x = 3x^2 + 2xy.$$

If the equation is exact, then

$$2xy + bx^2 = M_y = N_x = 3x^2 + 2xy \quad \text{so} \quad b = 3.$$

Setting $b = 3$ gives $(xy^2 + 3x^2y)\,dx + (x + y)x^2\,dy = 0$. Now I'll solve the equation.

I need to find f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$. Integrate the first equation with respect to x:

$$f = \int (xy^2 + 3x^2y)\,dx = \frac{1}{2}x^2y^2 + x^3y + h(y).$$

Differentiate f with respect to y and set the result equal to N:

$$(x+y)x^2 = \frac{\partial f}{\partial y} = x^2 y + x^3 + \frac{dh}{dy}.$$

Therefore,

$$\frac{dh}{dy} = 0 \quad \text{so} \quad h = 0.$$

Hence, $f = \frac{1}{2}x^2y^2 + x^3y$. The solution is

$$\frac{1}{2}x^2y^2 + x^3y = C. \square$$

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20. Show that the equation $\left(\frac{\sin y}{y} - e^{-x}\sin x\right) dx + \left(\frac{\cos y + 2e^{-x}\cos x}{y}\right) dy = 0$ is not exact, but becomes exact when multiplied by the integrating fractor $\mu(x,y) = ye^x$. Then solve the equation.

$$M_y = \frac{\cos y}{y} - \frac{\sin y}{y^2} \quad \text{and} \quad N_x = \frac{-2e^{-x}\cos x - 2e^{-x}\sin x}{y}.$$

The equation is not exact.
Multiply by $\mu = ye^x$:

$$(e^x \sin y - 2y\sin x)\,dx + (e^x\cos y + 2\cos x)\,dy = 0.$$

$$M_y = e^x\cos y - 2\sin x \quad \text{and} \quad N_x = e^x\cos y - 2\sin x,$$

so the equation is exact.

I need to find f such that $\frac{\partial f}{\partial x} = M$ and $\frac{\partial f}{\partial y} = N$. Integrate the first equation with respect to x:

$$f = \int (e^x\sin y - 2y\sin x)\,dx = e^x\sin y + 2y\cos x + h(y).$$

Differentiate f with respect to y and set the result equal to N:

$$e^x\cos y + 2\cos x = \frac{\partial f}{\partial y} = e^x\cos y + 2\cos x + \frac{dh}{dy}.$$

Therefore,

$$\frac{dh}{dy} = 0 \quad \text{so} \quad h = 0.$$

Hence, $f = e^x\sin y + 2y\cos x$. The solution is

$$e^x\sin y + 2y\cos x = C. \square$$

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25. Find an integrating factor, and solve the equation $(3x^2y + 2xy + y^3)\,dx + (x^2+y^2)\,dy = 0$.

$$M_y - N_x = 3x^2 + 2x + 3y^2 - 2x = 3x^2 + 3y^2, \quad \text{so} \quad \frac{M_y - N_x}{N} = 3.$$

Since $\dfrac{M_y - N_x}{N}$ is a function of x (in fact, it's constant), I'll use equation (26) on page 79 to compute an integrating factor:

$$\mu = \exp \int 3\, dx = e^{3x}.$$

Multiply by the integrating factor:

$$(3x^2 y + 2xy + y^3)e^{3x}\, dx + (x^2 + y^2)e^{3x}\, dy = 0.$$

Check:

$$M_y = 3x^2 e^{3x} + 2xe^{3x} + 3y^2 e^{3x} \quad \text{and} \quad N_x = 2xe^{3x} + 3x^2 e^{3x} + 3y^2 e^{3x},$$

so the equation is exact.

I need to find f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$. Integrate the first equation with respect to x:

$$f = \int (3x^2 ye^{3x} + 2xye^{3x} + y^3 e^{3x})\, dx =$$

$$x^2 ye^{3x} - \frac{2}{3}xye^{3x} + \frac{2}{9}ye^{3x} + \frac{2}{3}xye^{3x} - \frac{2}{9}ye^{3x} + \frac{1}{3}y^3 e^{3x} + h(y) =$$

$$x^2 ye^{3x} + \frac{1}{3}y^3 e^{3x} + h(y).$$

Differentiate f with respect to y and set the result equal to N:

$$(x^2 + y^2)e^{3x} = \frac{\partial f}{\partial y} = x^2 e^{3x} + y^2 e^{3x} + \frac{dh}{dy}.$$

Therefore,

$$\frac{dh}{dy} = 0 \quad \text{so} \quad h = 0.$$

Hence, $f = x^2 ye^{3x} + \dfrac{1}{3}y^3 e^{3x}$. The solution is

$$x^2 ye^{3x} + \frac{1}{3}y^3 e^{3x} = C.$$

Here's the work for the integrals:

$$\begin{array}{cccc}
 & \dfrac{d}{dx} & & \int\, dx \\
+ & x^2 & & e^{3x} \\
 & & \searrow & \\
- & 2x & & \dfrac{1}{3}e^{3x} \\
 & & \searrow & \\
+ & 2 & & \dfrac{1}{9}e^{3x} \\
 & & \searrow & \\
- & 0 & \rightarrow & \dfrac{1}{27}e^{3x}
\end{array}$$

$$\int x^2 e^{3x}\, dx = \frac{1}{3}x^2 e^{3x} - \frac{2}{9}xe^{3x} + \frac{2}{27}e^{3x} + C.$$

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x & & e^{3x} \\ & & \searrow & \\ - & 1 & & \dfrac{1}{3}e^{3x} \\ & & \searrow & \\ - & 0 & & \dfrac{1}{9}e^{3x} \end{array}$$

$$\int xe^{3x}\,dx = \frac{1}{3}xe^{3x} - \frac{1}{9}e^{3x} + C. \square$$

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28. Find an integrating factor, and solve the equation $y\,dx + (2xy - e^{-2y}\,dy = 0$.

$$N_x - M_y = 2y - 1, \quad \text{so} \quad \frac{M_y - N_x}{M} = \frac{2y-1}{y} = 2 - \frac{1}{y}.$$

Since $\dfrac{M_y - N_x}{M}$ is a function of y I'll use the method of Problem 23 to compute an integrating factor:

$$\mu = \exp \int \left(2 - \frac{1}{y}\right)\,dy = \frac{e^{2y}}{y}.$$

Multiply by the integrating factor:

$$e^{2y}\,dx + \left(2xe^{2y} - \frac{1}{y}\right)\,dy = 0.$$

Check:

$$M_y = 2e^2y \quad \text{and} \quad N_x = 2e^2y,$$

so the equation is exact.

I need to find f such that $\dfrac{\partial f}{\partial x} = M$ and $\dfrac{\partial f}{\partial y} = N$. Integrate the first equation with respect to x:

$$f = \int e^{2y}\,dx = xe^{2y} + h(y).$$

Differentiate f with respect to y and set the result equal to N:

$$2xe^{2y} - \frac{1}{y} = \frac{\partial f}{\partial y} = 2xe^{2y} + \frac{dh}{dy}.$$

Therefore,

$$\frac{dh}{dy} = -\frac{1}{y} \quad \text{so} \quad h = -\ln|y|.$$

Hence, $f = xe^{2y} - \ln|y|$. The solution is

$$xe^{2y} - \ln|y| = C. \square$$

"From a little distance one can perceive an order in what at the time seemed confusion." - F. Scott Fitzgerald