Math 228
2-2-1994

Solutions to Problem Set 7.

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1. Show that the equation $\dfrac{dy}{dx} = \dfrac{x+y}{x}$ is homogeneous, and find its solution.

Express the right side as a function of $\dfrac{y}{x}$:

$$\frac{dy}{dx} = \frac{x+y}{x} = 1 + \frac{y}{x}.$$

Make the substitution $y = xv$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$:

$$v + x\frac{dv}{dx} = 1 + v.$$

Separate variables:

$$\int dv = \int \frac{dx}{x}, \quad v = \ln|x| + C.$$

Put y back:

$$\frac{y}{x} = \ln|x| + C, \quad y = x\ln|x| + Cx. \square$$

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5. Show that the equation $\dfrac{dy}{dx} = \dfrac{4y-3x}{2x-y}$ is homogeneous, and find its solution.

Express the right side as a function of $\dfrac{y}{x}$:

$$\frac{dy}{dx} = \frac{4y-3x}{2x-y} = \frac{4\dfrac{y}{x} - 3}{2 - \dfrac{y}{x}}.$$

Make the substitution $y = xv$, $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$:

$$v + x\frac{dv}{dx} = \frac{4v-3}{2-v}.$$

Separate variables:

$$x\frac{dv}{dx} = \frac{4v-3}{2-v} - v = \frac{v^2+2v-3}{2-v}, \quad \int \frac{2-v}{v^2+2v-3}\,dv = \int \frac{dx}{x}.$$

Use partial fractions to do the integral on the left:

$$\frac{2-v}{(v+3)(v-1)} = \frac{A}{v+3} + \frac{B}{v-1}$$

$$2 - v = A(v-1) + B(v+3)$$

Setting $v = 1$ gives $1 = 4B$, so $B = \frac{1}{4}$. Setting $v = -3$ gives get $5 = -4A$, so $A = -\frac{5}{4}$. Therefore,

$$\frac{2-v}{v^2+2v-3} = -\frac{5}{4}\frac{1}{v+3} + \frac{1}{4}\frac{1}{v-1}.$$

Integrate:

$$\int \left(-\frac{5}{4}\frac{1}{v+3} + \frac{1}{4}\frac{1}{v-1}\right) dv = \int \frac{dx}{x}$$

$$-\frac{5}{4}\ln|v+3| + \frac{1}{4}\ln|v-1| = \ln|x| + C$$

Combine the logs on the left, then exponentiate to remove all the logs:

$$\frac{1}{4}\ln\left|\frac{v-1}{(v+3)^5}\right| = \ln|x| + C$$

$$\ln\left|\frac{v-1}{(v+3)^5}\right| = \ln x^4 + C_0$$

$$\left|\frac{v-1}{(v+3)^5}\right| = \exp(\ln x^4 + C_0) = C_1 x^4$$

Put y back:

$$\left|\frac{\frac{y}{x}-1}{(\frac{y}{x}+3)^5}\right| = C_1 x^4, \quad |y-x| = C_1|y+3x|^5. \ \square$$

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9. (a) Find the solution of the equation

$$\frac{dy}{dx} = \frac{2y-x}{2x-y}.$$

Express the right side as a function of $\frac{y}{x}$:

$$\frac{dy}{dx} = \frac{2y-x}{2x-y} = \frac{2\frac{y}{x}-1}{2-\frac{y}{x}}.$$

Make the substitution $y = xv$, $\frac{dy}{dx} = v + x\frac{dv}{dx}$:

$$v + x\frac{dv}{dx} = \frac{2v-1}{2-v}.$$

Separate variables:

$$x\frac{dv}{dx} = \frac{2v-1}{2-v} - v = \frac{v^2-1}{2-v}$$

$$\int \frac{2-v}{v^2-1}\,dv = \int \frac{dx}{x}$$

Use partial fractions to do the integral on the left:

$$\frac{2-v}{(v+1)(v-1)} = \frac{A}{v+1} + \frac{B}{v-1}$$

$$2 - v = A(v+1) + B(v-1)$$

Setting $v = 1$ gives $1 = 2A$, so $A = \dfrac{1}{2}$. Setting $v = -1$ gives $3 = -2B$, so $B = -\dfrac{3}{2}$. Therefore,

$$\frac{2-v}{v^2-1} = \frac{1}{2}\frac{1}{v-1} - \frac{3}{2}\frac{1}{v+1}.$$

Integrate:

$$\int \left(\frac{1}{2}\frac{1}{v-1} - \frac{3}{2}\frac{1}{v+1}\right)\, dv = \int \frac{dx}{x}$$

$$\frac{1}{2}\ln|v-1| - \frac{3}{2}\ln|v+1| = \ln|x| + C$$

Combine the logs on the left, then exponentiate to remove all the logs:

$$\frac{1}{2}\ln\left|\frac{v-1}{(v+1)^3}\right| = \ln|x| + C$$

$$\ln\left|\frac{v-1}{(v+1)^3}\right| = \ln x^2 + C_0$$

$$\left|\frac{v-1}{(v+1)^3}\right| = \exp(\ln x^2 + C_0) = C_1 x^2$$

Put y back:

$$\left|\frac{\frac{y}{x}-1}{(\frac{y}{x}+1)^3}\right| = C_1 x^2, \quad |y-x| = C_1|y+x|^3. \; \square$$

(b) Find the solution of the equation

$$\frac{dy}{dx} = \frac{2y - x + 5}{2x - y - 4}.$$

Change variables:

$$x = \bar{x} + h, \quad y = \bar{y} + k, \quad \text{so} \quad \frac{dy}{dx} = \frac{\dfrac{dy}{d\bar{x}}}{\dfrac{dx}{d\bar{x}}} = \frac{dy}{d\bar{x}} = \frac{dy}{d\bar{y}}\frac{d\bar{y}}{d\bar{x}} = \frac{d\bar{y}}{d\bar{x}}.$$

The right side of the original differential equation becomes

$$\frac{2(\bar{x}+h) - (\bar{y}+k) + 5}{2(\bar{x}+h) - (\bar{y}+k) - 4} = \frac{2\bar{y} - \bar{x} + (-h + 2k + 5)}{2\bar{x} - \bar{y} + (2h - k - 4)}.$$

The idea is to find values for h and k which make the constant terms vanish. This reduces the problem to part (a).

$$-h + 2k = -5 \quad \text{and} \quad -2h + 4h = -10 \quad \text{give} \quad h = 1, \quad k = -3.$$

The substitution is

$$x = \bar{x} + 1, \quad y = \bar{y} - 2.$$

The differential equation becomes

$$\frac{d\bar{y}}{d\bar{x}} = \frac{2\bar{y} - \bar{x}}{2\bar{x} - \bar{y}}.$$

By part (a), the solution is:

$$|\bar{y}-\bar{x}| = C_1|(\bar{y}+\bar{x})^3|.$$

Since $\bar{x} = x-1$ and $\bar{y} = y+2$, the answer is

$$|y-x+3| = C_1|(y+x+1)^3|.\ \square$$

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15. Show that the equation $y' = f(x,y)$ is homogeneous if $f(x,y)$ is such that

$$f(x,tx) = f(1,t)$$

where t is a real parameter. Use this fact to determine whether each of the following equations is homogeneous.

Suppose $f(x,tx) = f(1,t)$. To show that f is homogeneous, I need to show that $f(x,y)$ can be expressed as a function of $\dfrac{y}{x}$. Simply set $t = \dfrac{y}{x}$. Then

$$f(x,y) = f\left(1, \frac{y}{x}\right).$$

$f(x,y)$ is on the left, and the expression on the right is a function of $\dfrac{y}{x}$. Hence, f is homogeneous. $\square$

(a) $y' = \dfrac{x^3+xy+y^3}{x^2y+xy^2}$

It is clear by inspection that it isn't homogeneous, since there is a term of degree 2 (xy) in the numerator, whereas everything else has degree 3.

Using the criterion given in the problem,

$$f(x,xt) = \frac{x^3+x^2t+x^3t^3}{x^3t+x^3t^2} = \frac{x+t+xt^3}{xt+xt^2}, \quad \text{and} \quad f(1,t) = \frac{1+t+t^3}{t+t^2}.$$

Since $f(x,xt) \neq f(1,t)$, the equation is not homogeneous. (Essentially, this criterion amounts to checking whether $f(x,y)$ is a homogeneous function of degree 0.) $\square$

(b) $y' = \ln x - \ln y + \dfrac{x+y}{x-y}$

$$f(x,xt) = \ln x - \ln xt + \frac{x+xt}{x-xt} = \ln\frac{1}{t} + \frac{1+t}{1-t}, \quad \text{and} \quad f(1,t) = -\ln t + \frac{1+t}{1-t}.$$

Since $f(x,xt) = f(1,t)$, the equation is homogeneous. $\square$

"One might as well say he has sold when no one has bought as to say he has taught when no one has learned."
- JOHN DEWEY