Math 228
2-6-1994

Solutions to Problem Set 8.

[page 28]

28. $x^2y' + 2xy - y^3 = 0$, $x > 0$.

Rewrite the equation as $y' + \dfrac{2}{x}y = \dfrac{1}{x^2}y^3$. This is a Bernoulli equation with $n = 3$ (cf. Problem 27). Let $v = y^{1-3} = y^{-2}$, so $v' = -2y^{-3}y'$. Multiply the equation by $-2y^{-3}$:

$$-2y^{-3}y' - \frac{4}{x}y^{-2} = -\frac{2}{x^2}, \quad \text{so} \quad v' - \frac{4}{x}v = -\frac{2}{x^2}.$$

The integrating factor is

$$I = \exp \int -\frac{4}{x}\, dx = \exp -4\ln x = \exp \ln x^{-4} = x^{-4}.$$

Therefore,

$$vx^{-4} = \int -\frac{2}{x^6}\, dx = \frac{2}{5}x^{-5} + C.$$

The solution is

$$v = \frac{2}{5}\frac{1}{x} + Cx^4.$$

Now put y back:

$$y^{-2} = \frac{2}{5}\frac{1}{x} + Cx^4, \quad y^2 = \frac{5x}{2 + 5Cx^5}. \ \Box$$

---

29. $y' = ry - ky^2$, $r > 0$ and $k > 0$. This equation is important in population dynamics and is discussed in detail in Section 2.6.

Write the equation in the form $y' - ry = -ky^2$. It is a Bernoulli equation with $n = 2$. Let $v = y^{-1}$, so $v' = -y^{-2}y'$. Multiply $y' - ry = -ky^2$ by $-y^{-2}$ to obtain

$$-y^{-2}y' + \frac{r}{y} = k, \quad \text{or} \quad v' + rv = k.$$

You can treat this as separable or first-order linear in v. I will do the latter. Find an integrating factor:

$$I = \exp \int r\, dx = e^{rx}.$$

Therefore,

$$ve^{rx} = \int ke^{rx}\, dx = \frac{k}{r}e^{rx} + C.$$

The solution is

$$v = \frac{k}{r} + Ce^{-rx}.$$

Put y back:

$$\frac{1}{y} = \frac{k}{r} + Ce^{-rx}, \quad y = \frac{1}{\dfrac{k}{r} + Ce^{-rx}}.$$

(Of course, the original equation is the logistic equation.) □

[page 86]

36. Using the method of Problem 35 and the given particular solution, solve each of the following Riccati equations.

(a) $y' = 1 + x^2 - 2xy + y^2$, $y_1(x) = x$

Let $y = y_1 + \dfrac{1}{v} = x + \dfrac{1}{v}$. In the notation of Problem 35,

$$q_1 = 1 + x^2, \quad q_2 = -2x, \quad q_3 = 1.$$

Therefore,

$$\frac{dv}{dx} = 2xv - 2xv - 1 = -1, \quad v = -x + C, \quad y = x + \frac{1}{-x + C}. \ \square$$

(b) $y' = -\dfrac{1}{x^2} - \dfrac{y}{x} + y^2$, $y_1(x) = \dfrac{1}{x}$

Let $y = y_1 + \dfrac{1}{v} = \dfrac{1}{x} + \dfrac{1}{v}$. In the notation of Problem 35,

$$q_1 = -\frac{1}{x^2}, \quad q_2 = -\frac{1}{x}, \quad q_3 = 1.$$

Therefore,

$$\frac{dv}{dx} = -\left(-\frac{1}{x} + 2 \cdot \frac{1}{x}\right) v - 1, \quad \frac{dv}{dx} = -\frac{1}{x} v - 1, \quad \frac{dv}{dx} + \frac{1}{x} v = -1.$$

The integrating factor is

$$I = \exp \int \frac{1}{x}\, dx = x.$$

Hence,

$$xv = \int -x\, dx = -\frac{1}{2} x^2 + C.$$

The solution is

$$v = -\frac{1}{2} x + \frac{C}{x}.$$

Put y back:

$$y = \frac{1}{x} + \frac{1}{-\frac{1}{2} x + \frac{C}{x}} = \frac{1}{x} + \frac{2x}{C_0 - x^2}. \ \square$$

(c) $\dfrac{dy}{dx} = \dfrac{2(\cos x)^2 - (\sin x)^2 + y^2}{2 \cos x}$, $y_1(x) = \sin x$

Let $y = y_1 + \dfrac{1}{v} = \sin x + \dfrac{1}{v}$. In the notation of Problem 35,

$$q_1 = \cos x - \frac{(\sin x)^2}{2 \cos x}, \quad q_2 = 0, \quad q_3 = \frac{1}{2 \cos x}.$$

Therefore,

$$\frac{dv}{dx} = -\left(0 + 2 \cdot \frac{1}{2\cos x} \cdot \sin x\right) v - \frac{1}{2\cos x} = -\frac{\sin x}{\cos x} v - \frac{1}{2\cos x}.$$

Rewrite the equation in the form

$$\frac{dv}{dx} + \frac{\sin x}{\cos x} v = -\frac{1}{2\cos x}.$$

It is first-order linear. The integrating factor is

$$I = \exp \int \frac{\sin x}{\cos x}\, dx = \frac{1}{\cos x}.$$

Therefore,

$$v \cdot \frac{1}{\cos x} = \int -\frac{1}{2}(\sec x)^2\, dx = -\frac{1}{2}\tan x + C.$$

The solution is

$$v = -\frac{1}{2}\sin x + C\cos x.$$

Put y back:

$$y = \sin x + \frac{2}{C_0 \cos x - \sin x}. \ \square$$

"A taste for simplicity cannot endure for long." - Eugéne Delacroix