Math 228
3-3-1994

Phase portraits for 2-dimensional linear systems.

Name: ______________________________

This project is due by 3 p.m., Friday, March 25.

You will need to draw the direction fields for some systems of differential equations in this project. Begin by loading the following package:

```
Needs["Graphics`PlotField`"]
```

Remember that those are *backquotes* on either side of `PlotField`.

A lot of the work in this project involves matrices. Here is how to enter a matrix in *Mathematica*:

```
example = {{1, 2}, {5, -2}}
```

In mathematical notation, this is the matrix

$$\begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix}$$

You can use `MatrixForm` to display the matrix in mathematical form:

```
example // MatrixForm
```

When a matrix arises as the coefficient matrix of a linear system, the **eigenvalues** and **eigenvectors** can tell you a great deal about the solution curves. If A is a *square* matrix, λ is an **eigenvalue** of A with **eigenvector** $\vec{v}$ if

$$A \cdot \vec{v} = \lambda \vec{v}.$$

(The product on the right is matrix multiplication of the vector by the matrix.) For example, $\lambda = -4$ is an eigenvalue of the matrix above and $\vec{v} = \langle -2, 5 \rangle$ is a corresponding eigenvector:

$$\begin{bmatrix} 1 & 2 \\ 5 & -2 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \end{bmatrix} = \begin{bmatrix} 8 \\ -20 \end{bmatrix} = (-4) \cdot \begin{bmatrix} -2 \\ 5 \end{bmatrix}.$$

`Eigenvalues[]` computes the eigenvalues of a matrix:

```
Eigenvalues[example]
```

(This may fail for large matrices, due to problems inherent in finding roots of polynomials. In these cases, you may still be able to *approximate* the eigenvalues.)

`Eigenvectors[]` computes the eigenvectors:

```
Eigenvectors[example]
```

If you want to know which eigenvector goes with which eigenvalue, use `Eigensystem`. I will show the output for this command, so I can explain what it means.

```
Eigensystem[example]

    {{-4, 3}, {{-2, 5}, {1, 1}}}
```

Here's how to interpret the output. The first list `{-4, 3}` is the list of eigenvalues — they are $\lambda = -4$ and $\lambda = 3$. The second list is a list of the eigenvectors; it looks confusing because each of the two vectors is in turn a list!

The first eigenvector is $\langle -2, 5\rangle$ (`{-2, 5}`), and it belongs to the eigenvalue -4.

The second eigenvector is $\langle 1, 1\rangle$ (`{1, 1}`), and it belongs to the eigenvalue 3.

`Eigensystem` is a good command to use for the problems below, since you can get the eigenvalues and eigenvectors all at once.

The subject of this project is the qualitative behavior of solutions to linear systems of the form

$$\begin{aligned} \frac{dx}{dt} &= x' = ax + by, \\ \frac{dy}{dt} &= y' = cx + dy, \end{aligned}$$

where a, b, c, and d are constants.

The six examples below show the direction fields for some 2-dimensional linear systems. In each case, I've sketched in a few solution curves in gray. On the right side of each picture is the equation which produced the picture, and the name of the type of field in the picture.

For example, the first picture is the field for the system

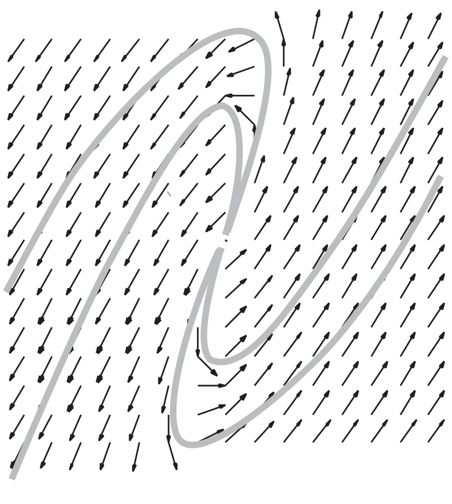
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

The first type of direction field is called a **spiral**.

It was plotted using the command:

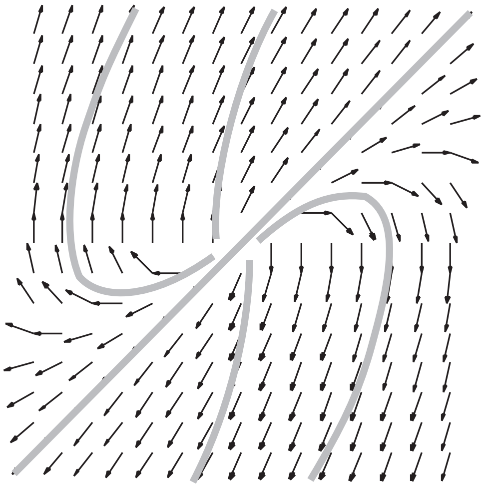
```
PlotVectorField[{3x - y, 5x - y}, {x, -1, 1},
    {y, -1, 1}, ScaleFunction->(0.75 &)]
```

Look through the pictures briefly to get an idea of the possibilities. Following the six examples are some problems for you to do.



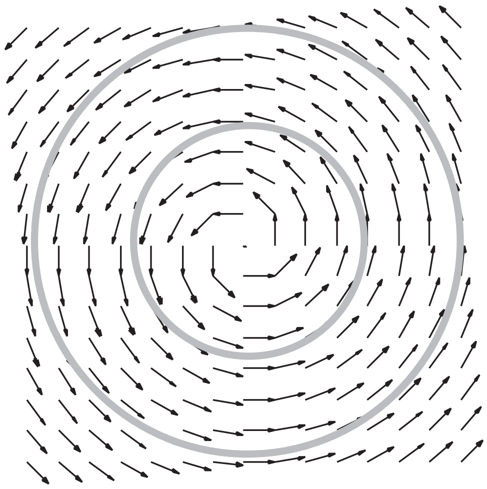
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 3 & -1 \\ 5 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Spiral



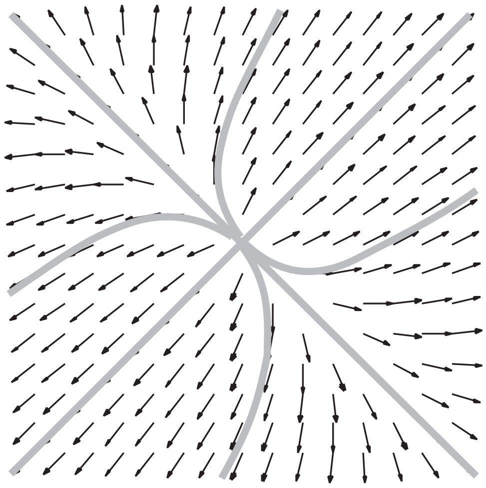
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Improper node



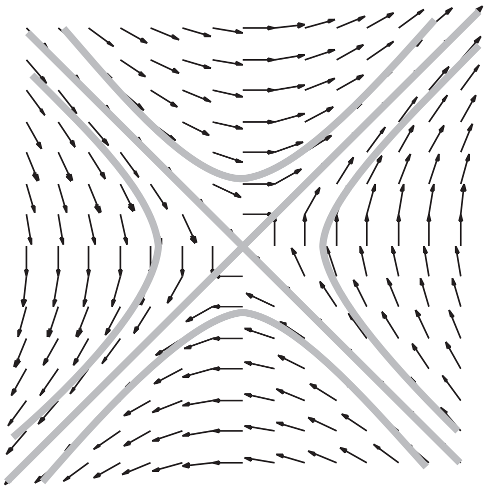
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Center



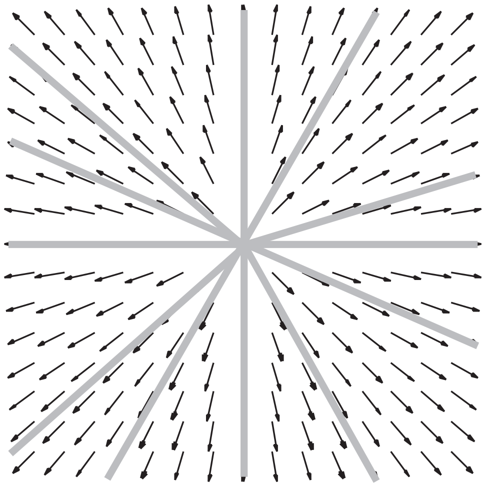
$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Improper node



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Saddle



$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Proper node

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1. Use `PlotVectorfield` to draw the direction field for the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 1 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In addition, use `Eigensystem` to find the eigenvalues and the eigenvectors. Finally, sketch in some solution curves, and say which of the six types of direction fields the picture resembles the most.

2. Use **`PlotVectorfield`** to draw the direction field for the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ -2 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In addition, use **`Eigensystem`** to find the eigenvalues and the eigenvectors. Finally, sketch in some solution curves, and say which of the six types of direction fields the picture resembles the most.

3. Use **`PlotVectorfield`** to draw the direction field for the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In addition, use **`Eigensystem`** to find the eigenvalues and the eigenvectors. Finally, sketch in some solution curves, and say which of the six types of direction fields the picture resembles the most.

4. Use **PlotVectorfield** to draw the direction field for the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 4 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In addition, use **Eigensystem** to find the eigenvalues and the eigenvectors. Finally, sketch in some solution curves, and say which of the six types of direction fields the picture resembles the most.

5. Use **PlotVectorfield** to draw the direction field for the system

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

In addition, use **Eigensystem** to find the eigenvalues and the eigenvectors. Finally, sketch in some solution curves, and say which of the six types of direction fields the picture resembles the most.