4-24-1994

Miscellaneous Review Problems for Test 3.

1. Find the terms of degree ≤ 5 for the series solution about $x_0 = 0$ for the equation

$$y'' + 2e^{x^3}y' + 2y = 0.$$

2. Find the series solution about $x_0 = 0$ for the equation

$$(x+1)y'' + 2xy' - y = 0.$$

3. Locate the critical points of the almost linear system

$$\frac{dx}{dt} = x(x+y-4),$$
$$\frac{dy}{dt} = y(x-1).$$

Find the corresponding linear system at each critical point, find the eigenvalues and eigenvectors, classify each critical point, and determine whether the critical point is stable, unstable, or asymptotically stable.

Solutions to the Miscellaneous Review Problems for Test 3.

1. Find the terms of degree ≤ 5 for the series solution about $x_0 = 0$ for the equation

$$y'' + 2e^{x^3}y' + 2y = 0.$$

Write

$$e^{x^3} = 1 + x^3 + \frac{x^6}{2!} + \frac{x^9}{3!} + \cdots.$$

I will use the first three terms. The differential equation becomes

$$y'' + 2\left(1 + x^3 + \frac{x^6}{2}\right)y' + 2y = 0.$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Substitute:

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=0}^{\infty} n a_n x^{n-1} + 2\sum_{n=0}^{\infty} n a_n x^{n+2} + \sum_{n=0}^{\infty} n a_n x^{n+5} + 2\sum_{n=0}^{\infty} a_n x^n = 0.$$

In the second sum, set $n - 1 = k - 2$, so $n = k - 1$. When $n = 0$, $k = 1$. The series becomes

$$2\sum_{n=0}^{\infty} n a_n x^{n-1} = 2\sum_{k=1}^{\infty}(k-1)a_{k-1}x^{k-2} = 2\sum_{n=1}^{\infty}(n-1)a_{n-1}x^{n-2}.$$

I obtained the second equality by replacing the dummy variable k with n.
In the third sum, set $n+2 = k-2$, so $n = k-4$. When $n = 0$, $k = 4$. The series becomes

$$2\sum_{n=0}^{\infty} na_n x^{n+2} = 2\sum_{k=4}^{\infty}(k-4)a_{k-4}x^{k-2} = 2\sum_{n=4}^{\infty}(n-4)a_{n-4}x^{n-2}.$$

I obtained the second equality by replacing the dummy variable k with n.
In the fourth sum, set $n+5 = k-2$, so $n = k-7$. When $n = 0$, $k = 7$. The series becomes

$$\sum_{n=0}^{\infty} na_n x^{n+5} = \sum_{k=7}^{\infty}(k-7)a_{k-7}x^{k-2} = \sum_{n=7}^{\infty}(n-7)a_{n-7}x^{n-2}.$$

I obtained the second equality by replacing the dummy variable k with n.
In the fifth sum, set $n = k-2$. When $n = 0$, $k = 2$. The series becomes

$$2\sum_{n=0}^{\infty} a_n x^n = 2\sum_{k=2}^{\infty} a_{k-2}x^{k-2} = 2\sum_{n=2}^{\infty} a_{n-2}x^{n-2}.$$

I obtained the second equality by replacing the dummy variable k with n.
Now the equation is

$$\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + 2\sum_{n=1}^{\infty}(n-1)a_{n-1}x^{n-2} + 2\sum_{n=4}^{\infty}(n-4)a_{n-4}x^{n-2} + \sum_{n=7}^{\infty}(n-7)a_{n-7}x^{n-2} + 2\sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0.$$

The sum of the coefficients for each power of x on the left must be 0. For $n = 0$ and $n = 1$, I get $0 = 0$.
For $n = 2$, I get

$$2a_2 + 2a_1 + a_0 = 0, \quad \text{so} \quad a_2 = -\frac{1}{2}a_0 - a_1.$$

For $n = 3$, I get

$$6a_3 + 4a_2 + a_1 = 0, \quad \text{so} \quad a_3 = \frac{1}{3}a_0 + \frac{1}{2}a_1.$$

(I substituted $a_2 = -\frac{1}{2}a_0 - a_1$ for a_2.)
For $n = 4$, I get

$$12a_4 + 6a_3 + a_2 = 0, \quad \text{so} \quad a_4 = -\frac{1}{8}a_0 - \frac{1}{6}a_1.$$

(I substituted for a_2 and a_3.)
For $n = 5$, I get

$$20a_5 + 8a_4 + 2a_1 + a_3 = 0, \quad \text{so} \quad a_5 = \frac{1}{30}a_0 - \frac{7}{120}a_1.$$

(I substituted for a_3 and a_4.)
The series is

$$y = a_0\left(1 - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{30}x^5 + \cdots\right) + a_1\left(x - x^2 + \frac{1}{2}x^3 - \frac{1}{6}x^4 - \frac{7}{120}x^5 + \cdots\right). \square$$

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2. Find the series solution about $x_0 = 0$ for the equation

$$(x+1)y'' + 2xy' - y = 0.$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2}.$$

Substitute:

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + +2 \sum_{n=0}^{\infty} n a_n x^n - \sum_{n=0}^{\infty} a_n x^n = 0.$$

In the first sum, let $n - 1 = k - 2$, so $n = k - 1$. When $n = 0$, $k = 1$, so the sum becomes

$$\sum_{n=0}^{\infty} n(n-1) a_n x^{n-1} = \sum_{k=1}^{\infty} (k-1)(k-2) a_{k-1} x^{k-2} = \sum_{n=1}^{\infty} (n-1)(n-2) a_{n-1} x^{n-2}.$$

I obtained the second equality by replacing the dummy variable k with n.
In the third and fourth sums, set $n = k - 2$. When $n = 0$, $k = 2$. The series become

$$2 \sum_{n=0}^{\infty} n a_n x^n = 2 \sum_{k=2}^{\infty} (k-2) a_{k-2} x^{k-2} = 2 \sum_{n=2}^{\infty} (n-2) a_{n-2} x^{n-2},$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=2}^{\infty} a_{k-2} x^{k-2} = \sum_{n=2}^{\infty} a_{n-2} x^{n-2}.$$

In each case, I obtained the second equality by replacing the dummy variable k with n.
Now the equation is

$$\sum_{n=1}^{\infty} (n-1)(n-2) a_{n-1} x^{n-2} + \sum_{n=0}^{\infty} n(n-1) a_n x^{n-2} + 2 \sum_{n=2}^{\infty} (n-2) a_{n-2} x^{n-2} - \sum_{n=2}^{\infty} a_{n-2} x^{n-2} = 0.$$

The sum of the coefficients for each power of x on the left must be 0. For $n = 0$ and $n = 1$, I get $0 = 0$.
For $n \geq 2$, I get

$$(n-1)(n-2) a_{n-1} + n(n-1) a_n + 2(n-2) a_{n-2} - a_{n-2} = 0, \quad \text{or} \quad a_n = -\frac{(2n-5) a_{n-2} + (n-1)(n-2) a_{n-1}}{n(n-1)}.$$

The series solution is

$$y = a_0 \left(1 + \frac{1}{2}x^2 - \frac{1}{6}x^3 - \frac{1}{24}x^4 + \frac{1}{15}x^5 - \cdots\right) + a_1 \left(x - \frac{1}{6}x^3 + \frac{1}{12}x^4 - \frac{1}{120}x^5 - \frac{1}{72}x^6 + \cdots\right). \square$$

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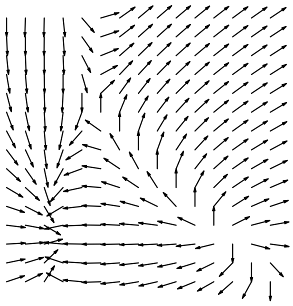
3. Locate the critical points of the almost linear system

$$\frac{dx}{dt} = x(x + y - 4),$$
$$\frac{dy}{dt} = y(x - 1).$$

Find the corresponding linear system at each critical point, find the eigenvalues and eigenvectors, classify each critical point, and determine whether the critical point is stable, unstable, or asymptotically stable.

First, I'll find the critical points:

$$\begin{array}{ccccc}
 & & x(x+y-4)=0 & & \\
 & \swarrow & & \searrow & \\
x=0 & & & & y=4-x \\
y(x-1)=0 & & & & y(x-1)=0 \\
-y=0 & & & & (4-x)(x-1)=0 \\
y=0 & & & \swarrow & \downarrow \\
(0,0) & & x=4 & & x=1 \\
 & & y=0 & & y=3 \\
 & & (4,0) & & (1,3)
\end{array}$$



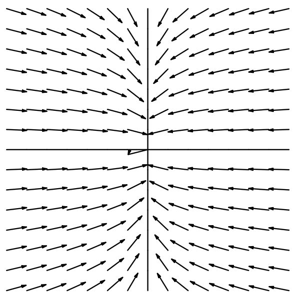
The matrix of partial derivatives is

$$A = \begin{bmatrix} 2x+y-4 & x \\ y & x-1 \end{bmatrix}.$$

At $(0,0)$,

$$A = \begin{bmatrix} -4 & 0 \\ 0 & -1 \end{bmatrix}.$$

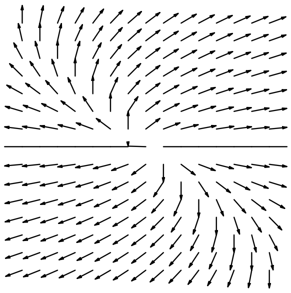
$\det = 4$ and $\operatorname{tr} = 5$, so $\det < \dfrac{1}{4}\operatorname{tr}^2$. The point is an asymptotically stable proper node.



The eigenvalues are -4 and -1; the corresponding eigenvectors are $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$.
At $(4,0)$,

$$A = \begin{bmatrix} 4 & 4 \\ 0 & 3 \end{bmatrix}.$$

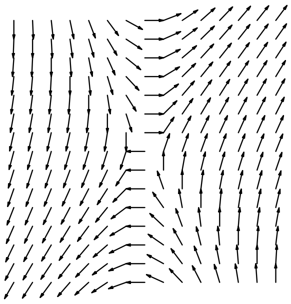
$\det = 12$ and $\operatorname{tr} = 7$, so $\det < \dfrac{1}{4}\operatorname{tr}^2$. The point is an unstable improper node.



The eigenvalues are 4 and 3; the corresponding eigenvectors are $\langle 1, 0\rangle$ and $\langle 4, -1\rangle$.
Finally, at $(1, 3)$,

$$A = \begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}.$$

$\det = -3$, so the point is a saddle, and it's unstable.



The eigenvalues are $\dfrac{1}{2} \pm \dfrac{\sqrt{13}}{2}$; the corresponding eigenvectors are $\left\langle 1, -\dfrac{1}{2} + \dfrac{\sqrt{13}}{2}\right\rangle$ and $\left\langle 1, -\dfrac{1}{2} - \dfrac{\sqrt{13}}{2}\right\rangle$.

$\square$