Math 228
4-28-1994

Solutions to Test 3.

If you want a problem regraded, *do not make any marks on or changes to your exam.* Write a note *on a separate piece of paper* and turn it back in with your exam at the end of class. You must ask for any regrades at the time you get your exam back.

1. Locate the critical points of the almost linear system

$$\frac{dx}{dt} = x(x+y-2),$$
$$\frac{dy}{dt} = (x^2+1)y.$$

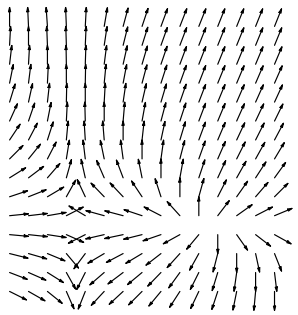
For each critical point, find the corresponding linear system, and find the eigenvalues and eigenvectors of the corresponding linear system. Classify the critical point, and determine whether it is stable, unstable, or asymptotically stable.

First, I'll find the critical points:

$$x(x+y-2) = 0$$

$$\swarrow \qquad \searrow$$

$$\begin{array}{c} x = 0 \\ (x^2+1)y = 0 \\ y = 0 \\ (0,0) \end{array} \qquad\qquad \begin{array}{c} x+y = 2 \\ (x^2+1)y = 0 \\ y = 0 \\ x = 2 \\ (2,0) \end{array}$$



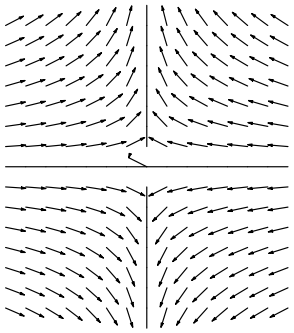
The matrix of partial derivatives is

$$A = \begin{bmatrix} 2x+y-2 & x \\ 2xy & x^2+1 \end{bmatrix}.$$

At $(0,0)$,

$$A = \begin{bmatrix} -2 & 0 \\ 0 & 1 \end{bmatrix}.$$

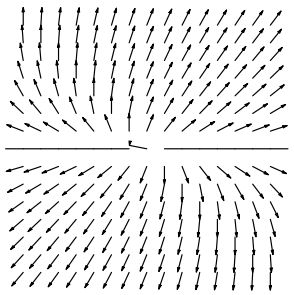
$\det = -2$, so the point is a saddle, and it's unstable.



The eigenvalues are -2 and 1; the corresponding eigenvectors are $\langle 1, 0\rangle$ and $\langle 0, 1\rangle$.
At $(2, 0)$,

$$A = \begin{bmatrix} 2 & 2 \\ 0 & 5 \end{bmatrix}.$$

$\det = 10$, $\mathrm{tr} = 7$, so $\det < \frac{1}{4}\,\mathrm{tr}^2$. The point is an unstable improper node.



The eigenvalues are 2 and 5; the corresponding eigenvectors are $\langle 1, 0\rangle$ and $\langle 2, 3\rangle$. □

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2. Solve the linear system

$$\vec{x}\,' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix} \vec{x}.$$

The characteristic polynomial is $\lambda^2 - 4\lambda + 5$. The eigenvalues are $\lambda = 2 \pm i$.
For $\lambda = 2 + i$,

$$A - (2+i)I = \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix}$$

Throw out the second row. I want a and b such that

$$(-i) \cdot a + 1 \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 1, i\rangle$.

Expand the solution for this eigenvalue and eigenvector:

$$e^{(2+i)t}\begin{bmatrix}1\\ i\end{bmatrix} = e^{2t}(\cos t + i\sin t)\begin{bmatrix}1\\ i\end{bmatrix} = e^{2t}\begin{bmatrix}\cos t + i\sin t\\ -\sin t + i\cos t\end{bmatrix}.$$

Take real and imaginary parts:

$$\operatorname{re} e^{(2+i)t}\begin{bmatrix}1\\ i\end{bmatrix} = e^{2t}\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix},$$

$$\operatorname{im} e^{(2+i)t}\begin{bmatrix}1\\ i\end{bmatrix} = e^{2t}\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}.$$

The general solution is

$$\vec{x} = e^{2t}\left(c_1\begin{bmatrix}\cos t\\ -\sin t\end{bmatrix} + c_2\begin{bmatrix}\sin t\\ \cos t\end{bmatrix}\right).$$

You could have done the problem using matrix exponentials, though I don't recommend doing it that way. In that case, the answer would be

$$\vec{x} = e^{2t}\begin{bmatrix}\cos t & \sin t\\ -\sin t & \cos t\end{bmatrix}\vec{c}. \square$$

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3. Find the series solution about $x_0 = 0$ for the equation

$$(x^2+4)y'' - 2xy' + y = 0.$$

Let

$$y = \sum_{n=0}^{\infty} a_n x^n.$$

The derivatives are

$$y' = \sum_{n=0}^{\infty} n a_n x^{n-1}, \quad y'' = \sum_{n=0}^{\infty} n(n-1)a_n x^{n-2}.$$

Substitute:

$$4\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=0}^{\infty} n(n-1)a_n x^n - 2\sum_{n=0}^{\infty} n a_n x^n + \sum_{n=0}^{\infty} a_n x^n = 0.$$

In the last three series, set $n = k-2$. When $n = 0$, $k = 2$, so the series become

$$\sum_{n=0}^{\infty} n(n-1)a_n x^n = \sum_{k=2}^{\infty}(k-2)(k-3)a_{k-2}x^{k-2} = \sum_{n=2}^{\infty}(n-2)(n-3)a_{n-2}x^{n-2},$$

$$2\sum_{n=0}^{\infty} n a_n x^n = 2\sum_{k=2}^{\infty}(k-2)a_{k-2}x^{k-2} = 2\sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2},$$

$$\sum_{n=0}^{\infty} a_n x^n = \sum_{k=2}^{\infty} a_{k-2}x^{k-2} = \sum_{n=2}^{\infty} a_{n-2}x^{n-2}.$$

In each case, I obtained the second equality by replacing the dummy variable k with n.

The differential equation is now

$$4\sum_{n=0}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=2}^{\infty}(n-2)(n-3)a_{n-2}x^{n-2} - 2\sum_{n=2}^{\infty}(n-2)a_{n-2}x^{n-2} + \sum_{n=2}^{\infty} a_{n-2}x^{n-2} = 0.$$

The sum of the coefficients for each power of x on the left must be 0. For $n = 0$ and $n = 1$, I get $0 = 0$. For $n \geq 2$, I get

$$4n(n-1)a_n + (n^2 - 7n + 11)a_{n-2} = 0, \quad a_n = -\frac{n^2 - 7n + 11}{4n(n-1)}a_{n-2}.$$

Plug in even numbers for n:

$$\begin{aligned} a_2 &= -\frac{1}{4 \cdot 2 \cdot 1}a_0, \\ a_4 &= -\frac{-1}{4 \cdot 4 \cdot 3}a_2, \\ &\vdots \\ a_{2n} &= -\frac{4n^2 - 14n + 11}{4 \cdot (2n)(2n-1)}a_{2n-2}. \end{aligned}$$

Multiply the equations together — that is, multiply the left sides, multiply the right sides, then set the two products equal:

$$a_2 a_4 \cdots a_{2n} = \left(-\frac{1}{4 \cdot 2 \cdot 1}\right)\left(-\frac{-1}{4 \cdot 4 \cdot 3}\right)\cdots\left(-\frac{4n^2 - 14n + 11}{4 \cdot (2n)(2n-1)}\right) a_0 a_2 \cdots a_{2n-2}.$$

Cancel the common terms a_2, a_4, ..., a_{2n-2} from both sides:

$$a_{2n} = \left(-\frac{1}{4 \cdot 2 \cdot 1}\right)\left(-\frac{-1}{4 \cdot 4 \cdot 3}\right)\cdots\left(-\frac{4n^2 - 14n + 11}{4 \cdot (2n)(2n-1)}\right) a_0.$$

There are n negative signs, giving a factor of $(-1)^n$. There are n 4's on the bottom, giving a factor of 4^n. The rest of the denominators of the fractions multiply to $(2n)!$. So

$$a_{2n} = (-1)^n \frac{1 \cdot (-1) \cdots (4n^2 - 14n + 11)}{4^n (2n)!} a_0.$$

Similarly, I can plug in odd numbers for n;

$$\begin{aligned} a_3 &= -\frac{-1}{4 \cdot 3 \cdot 2}a_1, \\ a_5 &= -\frac{1}{4 \cdot 5 \cdot 4}a_3, \\ &\vdots \\ a_{2n+1} &= -\frac{4n^2 - 10n + 5}{4 \cdot (2n+1)(2n)}a_{2n-1}. \end{aligned}$$

Multiply the equations together — that is, multiply the left sides, multiply the right sides, then set the two products equal:

$$a_3 a_5 \cdots a_{2n+1} = \left(-\frac{-1}{4 \cdot 3 \cdot 2}\right)\left(-\frac{1}{4 \cdot 5 \cdot 4}\right)\cdots\left(-\frac{4n^2 - 10n + 5}{4 \cdot (2n+1)(2n)}\right) a_1 a_3 \cdots a_{2n-1}.$$

Cancel the common factors a_3, a_5, ..., a_{2n-1} from both sides:

$$a_{2n+1} = \left(-\frac{-1}{4 \cdot 3 \cdot 2}\right)\left(-\frac{1}{4 \cdot 5 \cdot 4}\right) \cdots \left(-\frac{4n^2 - 10n + 5}{4 \cdot (2n+1)(2n)}\right) a_1.$$

There are n negative signs, giving a factor of $(-1)^n$. There are n 4's on the bottom, giving a factor of 4^n. The rest of the denominators of the fractions multiply to $(2n+1)!$. So

$$a_{2n+1} = (-1)^n \frac{(-1) \cdot 1 \cdots (4n^2 - 10n + 5)}{4^n (2n+1)!} a_1.$$

The solution is

$$y = a_0 \left(1 + \sum_{n=1}^{\infty} (-1)^n \frac{1 \cdot (-1) \cdots (4n^2 - 14n + 11)}{4^n (2n)!} x^{2n}\right) +$$
$$a_1 \left(x + \sum_{n=1}^{\infty} (-1)^n \frac{(-1) \cdot 1 \cdots (4n^2 - 10n + 5)}{4^n (2n+1)!} x^{2n+1}\right). \square$$

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4. Find the eigenvalues and corresponding eigenvectors for

$$A = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 1 & -2 & 2 \end{bmatrix}.$$

The matrix is triangular, so the diagonal entries are the eigenvalues: $\lambda = 3$ (double) and $\lambda = 2$.
For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & -2 & -1 \end{bmatrix}$$

If $\langle a, b, c \rangle$ is an eigenvector, then $a - 2b - c = 0$, so $a = 2b + c$. Therefore, the eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 2b + c \\ b \\ c \end{bmatrix} = b \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + c \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Two independent eigenvectors are $\langle 2, 1, 0 \rangle$ and $\langle 1, 0, 1 \rangle$.
For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1 & -2 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

If $\langle a, b, c \rangle$ is an eigenvector, then $a = 0$ and $b = 0$. Therefore, the eigenvectors are

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ c \end{bmatrix}.$$

Thus, $\langle 0, 0, 1 \rangle$ is an eigenvector. $\square$

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5. Compute e^{At} for

$$A = \begin{bmatrix} 3 & 4 \\ 2 & -4 \end{bmatrix}.$$

The characteristic polynomial is

$$(3-\lambda)(-4-\lambda)-8=\lambda^2+\lambda-20=(\lambda+5)(\lambda-4).$$

The eigenvalues are $\lambda=-5$ and $\lambda=4$.
First, I'll compute the a_k's:

$$a_1=e^{-5t},$$

$$a_2=e^{-5t}\star e^{4t}=\int_0^t e^{-5(t-u)}e^{4u}\,du=e^{-5t}\int_0^t e^{9u}\,du=\frac{1}{9}e^{4t}-\frac{1}{9}e^{-5t}.$$

Next, I'll compute the B_k's:

$$B_1=I,\quad B_2=A+5I=\begin{bmatrix}8&4\\2&1\end{bmatrix}.$$

Therefore,

$$e^{At}=e^{-5t}\begin{bmatrix}1&0\\0&1\end{bmatrix}+\left(\frac{1}{9}e^{4t}-\frac{1}{9}e^{-5t}\right)\begin{bmatrix}8&4\\2&1\end{bmatrix}=\frac{1}{9}\begin{bmatrix}8e^{4t}+e^{-5t}&4e^{4t}-4e^{-5t}\\2e^{4t}-2e^{-5t}&e^{4t}+8e^{-5t}\end{bmatrix}.\ \square$$

"Men shut their doors against a setting sun." - SHAKESPEARE, *Timon of Athens*