Equivalence relations.

Definition. Let X be a set. $X \times X$ is the **Cartesian product** of X with itself. It consists of all ordered pairs (x_1, x_2) where $x_1, x_2 \in X$.

For example, if $X = \{1, 2, \text{Calvin}\}$, then

$$X \times X = \{(1,1), (1,2), (1, \text{Calvin}), (2,1), (2,2), (2, \text{Calvin}), (\text{Calvin}, 1), (\text{Calvin}, 2), (\text{Calvin}, \text{Calvin})\}.$$

Definition. A **relation** on X is a subset of $X \times X$. For example,

$$\{(1,2), (1, \text{Calvin}), (\text{Calvin}, 1), (\text{Calvin}, 2)\}$$

is a relation on the set $X = \{1, 2, \text{Calvin}\}$.

A better example might be the "greater than" relation on the set of integers $\mathbb{Z}$. This subset of $\mathbb{Z} \times \mathbb{Z}$ consists of all pairs (m, n) such that $m > n$. For example, $(10, -3)$ are in the set $(7, 1)$, while $(19, 35)$ is not.

Of course, you write $10 > -3$, $7 > 1$, and $19 \not> 35$ rather than listing the ordered pairs. It's customary to use **infix notation** when dealing with relations.

Thus, if I'm discussing a particular relation on a set, I'll often denote it by writing $a \sim b$. This means that the pair (a, b) is an element of the relation (considered as a subset of $X \times X$). When I do this, I'll often refer to "the relation $\sim$", even though a relation is really a set of ordered pairs.

(Why does this definition make sense in view of the way the word "relation" is used in everyday speech? Consider the following relation on the set of objects on the surface of the earth. (I'll take for granted that this set is well defined.) If a and b are objects, I will write $a \sim b$ if a is heavier than b. For example, if a is my copy of the book *Omeros* by Derek Walcott and b denotes the BIC pen in my pocket, then $a \sim b$.

You can completely specify this relation by letting X be the set of objects on the surface of the earth. Then consider the subset of $x \times X$ which consists of all pairs of objects (a, b), where a is heavier than b.)

I'm going to single out a special kind of relation. It is what you use when you say two things are "the same" in the context of a discussion.

Definition. An **equivalence relation** on a set X is a relation $\sim$ on X such that:

1. $x \sim x$ for all $x \in X$. (The relation is **reflexive**.)
2. If $x \sim y$, then $y \sim x$. (The relation is **symmetric.**)
3. If $x \sim y$ and $y \sim z$, then $x \sim z$. (The relation is **transitive**.)

Examples. 1. The "greater than" relation on the set of integers $\mathbb{Z}$is not an equivalence relation. It is not reflexive — $5 \not> 5$, for instance. It's also not symmetric: $7 > 3$ but $3 \not> 7$.

Rather trivially, the relation $=$ of equality is an equivalence relation on $\mathbb{Z}$. I'll check that the three conditions are satisfied.

If $n \in \mathbb{Z}$, then $n = n$. It's reflexive.

If $m, n \in \mathbb{Z}$ and $m = n$, then $n = m$. It's symmetric.

If $m, n, p \in \mathbb{Z}$, $m = n$, and $n = p$, then $m = p$. It's transitive.

Hence, it's an equivalence relation. $\square$

2. If $a, b \in \mathbb{Z}$ and n is a fixed positive integer, write $a \sim b$ if $n | (a - b)$. I'll show that this is an equivalence relation. Let $a, b, c \in \mathbb{Z}$.

$n | (a - a) = 0$, so $a \sim a$. It's reflexive.

Suppose $a \sim b$. That means $n | (a - b)$, Then surely $n | (b - a)$ — if n divides a number, it divides its negative — so $b \sim a$. It's symmetric.

Finally, suppose $a \sim b$ and $b \sim c$. This means that $n | (a - b)$ and $n | (b - c)$. Write

$$a - b = nx, \quad b - c = ny, \qquad \text{where } x, y \in \mathbb{Z}.$$

Then

$$a - c = (a - b) + (b - c) = nx + ny = n(x + y).$$

Therefore, $n|(a - c)$, so $a \sim c$. It's transitive.

This equivalence relation is called **congruence mod** n. Instead of writing $a \sim b$, write $a = b \pmod{n}$. For example, $3483 = 408696 \pmod{7109}$. □

Equivalence relations give rise to **partitions**. Here's an example before I give the definition. Consider the equivalence relation of congruence mod 3 on $\mathbb{Z}$. The integers break up into three *disjoint* sets:

$$\{\ldots, -6, -3, 0, 3, 6, \ldots\}, \{\ldots, -5, -2, 1, 4, 7, \ldots\}, \{\ldots, -4, -1, 2, 5, 8, \ldots\}.$$

All the elements of a given set are congruent mod 3, and no element in one set is congruent mod 3 to an element of another. The sets divide up the integers like three puzzle pieces. The three sets are called the **equivalence classes** corresponding to the equivalence relation.

In general, if $\sim$ is an equivalence relation on a set X and $x \in X$, the **equivalence class of** x consists of all the elements of X which are equivalent to x.

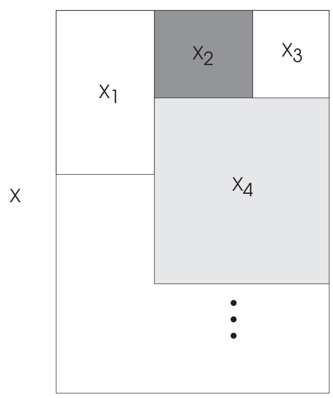
Definition. Let X be a set. A **partition** of X is a collection of subsets $\{X_i\}_{i \in I}$ of X such that:

1. $X = \bigcup_{i \in I} X_i$.
2. If $i, j \in I$ and $i \neq j$, then $X_i \cap X_j = \emptyset$.

(To refresh your memory: If A and B are sets, their **union** $A \cup B$ is the set whose elements are elements of A or B (or both). Their **intersection** $A \cap B$ is the set whose elements are the elements *common to* A and B. For example, if $A = \{1, 2, 3, 4\}$ and $B = \{2, 3, 4, \text{Calvin}\}$, then $A \cup B = \{1, 2, 3, 4, \text{Calvin}\}$, while $A \cap B = \{2, 3, 4\}$.)

Example. The four suits (spades, hearts, diamonds, clubs) partition a deck of playing cards. □

Thus, the elements of a partition are like the pieces of a jigsaw puzzle:



Here is how equivalence relations are related to partitions.

Theorem. Let X be a set. An equivalence relation $\sim$ on X gives rise to a partition of X into **equivalence classes**. Conversely, a partition of X gives rise to an equivalence relation on X whose equivalence classes are exactly the elements of the partition.

Proof. Suppose $\sim$ is an equivalence relation on X. If $x \in X$, let

$$S(x) = \{y \in X \mid y \sim x\}$$

denote the equivalence class of x. $x \sim x$, so $x \in S(x)$. Clearly, $X = \bigcup_{x \in X} S(x)$.

Now some of the $S(x)$'s may be identical; throw out the duplicates. This means that I have $S(x)$'s where $x \in Y$, and Y is a subset of X — and if $x, y \in Y$ and $x \neq y$, then $S(x) \neq S(y)$. Since I've just thrown out duplicates, I still have $X = \bigcup_{x \in Y} S(x)$. I will have a partition if I show that the remaining $S(x)$'s don't intersect.

Suppose $x, y \in Y$, $x \neq y$, but $z \in S(x) \cap S(y)$. I'll show that this gives a contradiction. By definition, $z \sim x$ and $z \sim y$, so by symmetry and transitivity, $x \sim y$.

Now I'll show $S(x) = S(y)$. *The standard way to show two sets are equal is to show each is contained in the other.* Suppose $w \in S(x)$. Then $w \sim x$, but $x \sim y$, so $w \sim y$, and $s \in S(y)$. This shows $S(x) \subset S(y)$. But the argument clearly works the other way around, so $S(y) \subset S(x)$. Hence, $S(x) = S(y)$.

Since I threw out all the duplicates earlier, this is a contradiction. Hence, there is no such z: $S(x) \cap S(y) = \emptyset$. This means that the $S(x)$'s for $x \in Y$ partition X.

Conversely, suppose $\{X_i\}_{i \in I}$ is a partition of X. Define a relation on X by saying $x \sim y$ if and only if $x, y \in X_i$ for some $i \in I$.

If $x \in X$, $x \in X_i$ for some i because $X = \bigcup_{i \in I} X_i$. To be silly, $x, x \in X_i$, so $x \sim x$. It's reflexive.

If $x \sim y$, then $x, y \in X_i$ for some i. Obviously, $y, x \in X_i$, so $y \sim x$. It's symmetric.

Finally, if $x \sim y$ and $y \sim z$, then $x, y \in X_i$ and $y, z \in X_j$ for some i and j. Now $y \in x_i \cap X_j$, but this can only happen if $X_i = X_j$. Then $x, z \in X_i$, so $x \sim z$. It's transitive, and hence it's an equivalence relation.

The equivalence classes of $\sim$ are exactly the X_i's, by construction. $\Box$

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