Math 307
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When does a set of cosets form a group?

Here is the multiplication table for D_3, the group of symmetries of a triangle. The subgroup $H = \{id, r_1, r_2\}$ has two cosets: H itself and the set $\{m_1, m_2, m_3\}$. Color the two cosets in the table below. (For example, leave H uncolored and color the elements of $\{m_1, m_2, m_3\}$.)

	id	r_1	r_2	m_1	m_2	m_3
id	id	r_1	r_2	m_1	m_2	m_3
r_1	r_1	r_2	id	m_3	m_1	m_2
r_2	r_2	id	r_1	m_2	m_3	m_1
m_1	m_1	m_2	m_3	id	r_1	r_2
m_2	m_2	m_3	m_1	r_2	id	r_1
m_3	m_3	m_1	m_2	r_1	r_2	id

Consider the product of two elements ab. The coloring shows that the coset containing the product depends only on the cosets containing a and b. Suppose ab is in the coset colored green. Take a' in the same coset as a and b' in the same coset as b. Then $a'b'$ will also be in the coset colored green.

Here is the same table rearranged to fit the subgroup $K = \{id, m_1\}$ and its cosets $r_1K = \{r_1, m_3\}$ and $r_2K = \{r_2, m_2\}$. Color the three cosets with different colors.

	id	m_1	r_1	m_3	r_2	m_2
id	id	m_1	r_1	m_3	r_2	m_2
m_1	m_1	id	m_2	r_2	m_3	r_1
r_1	r_1	m_3	r_2	m_2	id	m_1
m_3	m_3	r_1	m_1	id	m_2	r_2
r_2	r_2	m_2	id	m_1	r_1	m_3
m_2	m_2	r_2	m_3	r_1	m_1	id

In this case, the coset containing a product $a \cdot b$ depends on the particular elements a and b, not just on the cosets containing a and b. The coloring produces a table that is not arranged in nice "blocks" like the previous table. For example, $r_1 \cdot r_1 = r_2$, which is in the third coset. On the other hand, $m_3 \cdot m_3 = id$, which is in the first coset. You get different cosets, even though the factors in the two products are all in the second coset.

Conclusion: The subgroup H is **normal**; the subgroup K is not.