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Review Problems.

1. Let $G = \mathbb{R} - \mathbb{Q}$ be the set of irrational numbers. Is G a group under addition?

No. If blah is a putative identity, then (for instance) $\pi + \text{blah} = \pi$, which implies that blah $= 0$. However, 0 is rational, not irrational. Therefore, G does not have an identity element. $\square$

2. What are the subgroups of the Klein 4-group?

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

The order of a subgroup must divide the order of the group. Hence, there *could be* subgroups of order 1, 2, or 4. The subgroup of order 1 is $\{1\}$; the subgroup of order 4 is the whole group. A subgroup of order 2 must contain the identity and another element; by closure under inverses, the other element must be its own inverse. Hence, the subgroups of V are:

$$V, \{1, a\}, \{1, b\}, \{1, c\}, \{1\}. \square$$

3. (a) Let S and T be sets, $g : S \to T$ an injection, and $A, B \subset S$. Prove that $g(A \cap B) = g(A) \cap g(B)$.

I'll prove the inclusions $g(A \cap B) \subset g(A) \cap g(B)$ and $g(A) \cap g(B) \subset g(A \cap B)$.

First, $A \cap B \subset A$, so $g(A \cap B) \subset g(A)$. Likewise, $A \cap B \subset B$, so $g(A \cap B) \subset g(B)$. Therefore, $g(A \cap B) \subset g(A) \cap g(B)$.

Next, let $t \in g(A) \cap g(B)$. Then $t \in g(A)$, so $t = g(a)$ for some $a \in A$. Likewise, $t \in g(B)$, so $t = g(b)$ for some $b \in B$. Hence, $g(a) = t = g(b)$, so $a = b$ since g is 1-1. Therefore, $a = b \in A \cap B$, so $t = g(a) \in g(A \cap B)$. Hence, $g(A) \cap g(B) \subset g(A \cap B)$.

Therefore, $g(A \cap B) = g(A) \cap g(B)$. $\square$

(b) Give an example of sets S and T, a map $g : S \to T$, and subset $A, B \subset S$ such that $g(A \cap B) \neq g(A) \cap g(B)$.

Let $S = T = \mathbb{R}$, let $g : \mathbb{R} \to \mathbb{R}$ be given by $g(x) = x^2$, and let

$$A = \{x \in \mathbb{R} \mid x < 0\}, \qquad B = \{x \in \mathbb{R} \mid x > 0\}.$$

Since $A \cap B = \emptyset$, $g(A \cap B) = \emptyset$. However,

$$g(A) = g(B) = \{x \in \mathbb{R} \mid x > 0\}.$$

Hence, $g(A) \cap g(B) = \{x \in \mathbb{R} \mid x > 0\}$. Therefore, $g(A \cap B) \neq g(A) \cap g(B)$. $\square$

4. What elements generate the cyclic group $\mathbb{Z}_{10}$?

The order of the element $m \in \mathbb{Z}_{10}$ is $\dfrac{10}{(m, 10)}$, where $(m, 10)$ is the greatest common divisor of m and 10. If m is to generate $\mathbb{Z}_{10}$, it must have order 10. However, $\dfrac{10}{(m, 10)} = 10$ implies that $(m, 10) = 1$; that is, m is relatively prime to 10. The generators are $\{1, 3, 7, 9\}$. $\square$

5. Is the set

$$\left\{0, 1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$$

well-ordered?

No. The subset

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$$

does not have a smallest element. □

6. Suppose $m, n \in \mathbb{Z}$ and $(m, n) = 1$. Prove that $(m + n, m - n)$ is either 1 or 2.

If $d > 0$ and $d \mid m + n$ and $d \mid m - n$, then by adding and subtracting I find that

$$d \mid 2m \text{ and } d \mid 2n.$$

Therefore, $d \mid (2m, 2n) = 2(m, n) = 2$, so $d = 1$ or $d = 2$.

Both cases can occur. If $m = 7$ and $n = 2$, then $(m + n, m - n) = (9, 5) = 1$. If $m = 7$ and $n = 3$, then $(m + n, m - n) = (10, 4) = 2$. □

7. What is the order of

$$(1483)(2734) \in S_8?$$

Is this permutation even or odd?

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 4 & 2 & 1 & 8 & 5 & 6 & 7 & 3 \\ 2 & 7 & 1 & 8 & 5 & 6 & 3 & 4 \end{pmatrix}$$

The product is therefore $(1273)(48)$. It has order 4.

Since

$$(1273)(48) = (12)(17)(13)(48),$$

the permutation is even. □

8. Let $G = \mathbb{R} - \{-1\}$. For $a, b \in G$, define

$$a \star b = a + b + ab.$$

(The operations on the right are ordinary arithmetic operations.) Show that G is a group.

The operation is associative:

$$a \star (b \star c) = a + b + c + ab + ac + bc + abc = (a \star b) \star c.$$

0 is the identity:

$$0 \star a = a = a \star 0.$$

Finally, $a^{-1} = -\dfrac{a}{a+1}$. For

$$a \star \left(-\frac{a}{a+1}\right) = a - \frac{a}{a+1} - \frac{a^2}{a+1} = \frac{a^2 + a - a - a^2}{a+1} = 0,$$

and likewise $\left(-\dfrac{a}{a+1}\right) \star a = 0$.

Hence, G is a group. □

9. Let X be an infinite set, and let $H \subset S_X$ be the set of permutations which only move finitely many elements. Show that $H < S_X$.

If $\sigma, \tau \in H$, then σ and τ each move only finitely many elements, so the composite $\sigma \cdot \tau$ only moves finitely many elements. Hence, $\sigma \cdot \tau \in H$.

The identity permutation is in H, since it moves no elements.

If σ moves only finitely many elements, its inverse only moves the same elements as σ. Hence, σ^{-1} is in H.

Therefore, H is a subgroup. $\square$