Coordinates and Change of basis.

Let V be a vector space and let $\mathcal{B}$ be a basis for V. Every vector $v \in V$ can be uniquely expressed as a linear combination of elements of $\mathcal{B}$:

$$v = a_1 v_1 + \cdots + a_n v_n, \qquad v_i \in \mathcal{B}, \quad a_i \in F.$$

(Let me remind you why this is true. Since a basis spans, every $v \in V$ can be written in this way. On the other hand, if $a_1 v_1 + \cdots + a_n v_n = a_1' v_1 + \cdots + a_n' v_n$ are two ways of writing a given vector, then $(a_1 - a_1')v_1 + \cdots (a_n - a_n')v_n = 0$, and by independence $a_1 - a_1' = 0$, ..., $a_n - a_n' = 0$ — that is, $a_1 = a_1'$, ..., $a_n = a_n'$. So the representation of a vector in this way is unique.)

Consider the situation where $\mathcal{B}$ is a **finite ordered basis** — that is, fix a numbering $v_1, \ldots, v_n$ of the elements of $\mathcal{B}$. If $v = a_1 v_1 + \cdots + a_n v_n$, the ordered list of coefficients $\langle a_1, \ldots, a_n \rangle$ is uniquely associated with v. The $\{a_i\}$ are the **components** of v with respect to the (ordered) basis $\mathcal{B}$; I will use the notation

$$v = \langle a_1, \ldots, a_n \rangle_{\mathcal{B}}.$$

It is easy to confuse a vector with the *representation* of the vector in terms of its components relative to a basis. This confusion arises because representation of vectors which is most familiar is that of a vector as an ordinary n-tuple in $\mathbb{R}^n$:

$$\mathbb{R}^n = \{\langle a_1, \ldots, a_n \rangle \mid a_i \in \mathbb{R}\}.$$

This amounts to identifying the elements of $\mathbb{R}^n$ with their representation relative to the **standard basis**

$$\begin{aligned} e_1 &= \langle 1, 0, 0, \ldots, 0 \rangle, \\ e_2 &= \langle 0, 1, 0, \ldots, 0 \rangle, \\ &\vdots \\ e_n &= \langle 0, 0, 0, \ldots, 1 \rangle. \end{aligned}$$

Example. (a) Show that

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} \right\}$$

is a basis for $\mathbb{R}^3$.

These are three vectors in $\mathbb{R}^3$, which has dimension 3. Hence, it suffices to check that they're independent. Form the matrix with the elements of $\mathcal{B}$ as its rows and row reduce:

$$\begin{bmatrix} 1 & 0 & 1 \\ 2 & -1 & 1 \\ 3 & 1 & 0 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The vectors are independent. Three independent vectors in $\mathbb{R}^3$ must form a basis. □

(b) Find the components of $\langle 15, -1, 2 \rangle$ relative to $\mathcal{B}$.

I must find numbers a, b, and c such that

$$\begin{bmatrix} 15 \\ -1 \\ 2 \end{bmatrix} = a \cdot \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + b \cdot \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} + c \cdot \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}.$$

This is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 2 \end{bmatrix}.$$

Set up the matrix for the system and row reduce to solve:

$$\begin{bmatrix} 1 & 2 & 3 & 15 \\ 0 & -1 & 1 & -1 \\ 1 & 1 & 0 & 2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & 3 \end{bmatrix}$$

This says $a = -2$, $b = 4$, and $c = 3$. Therefore, $\langle 15, -1, 2 \rangle = \langle -2, 4, 3 \rangle_{\mathcal{B}}$. □

(c) Write $\langle 7, -2, 2 \rangle_{\mathcal{B}}$ in terms of the standard basis.

I'll write $v_{\mathcal{B}}$ for v relative to $\mathcal{B}$ and v_{std} for v relative to the standard basis. The matrix equation in (b)

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 15 \\ -1 \\ 2 \end{bmatrix}$$

says

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} v_{\mathcal{B}} = v_{\text{std}}.$$

In (b), I knew v_{std} and I wanted $v_{\mathcal{B}}$; this time it's the other way around. So I simply put $\langle 7, -2, 2 \rangle_{\mathcal{B}}$ into the $v_{\mathcal{B}}$ spot and multiply:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 7 \\ -2 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 4 \\ 5 \end{bmatrix}. \ \square$$

Let me generalize the observation I made in (c).

- If $\mathcal{B}$ is a basis for $\mathbb{R}^n$ — whose elements are written in terms of the standard basis, of course — and M is the matrix whose columns are the vectors in $\mathcal{B}$, then left multiplication by M *translates* vectors from $\mathcal{B}$ to the standard basis.

$$M v_{\mathcal{B}} = v_{\text{std}}.$$

The **inverse** of a square matrix M is a matrix M^{-1} such that $MM^{-1} = M^{-1}M = I$, where I is the identity matrix. If I multiply the last equation on the left by M^{-1}, I get

$$M^{-1} M v_{\mathcal{B}} = M^{-1} v_{\text{std}}, \quad \text{or} \quad v_{\mathcal{B}} = M^{-1} v_{\text{std}}.$$

In words, this means:

- Left multiplication by M^{-1} *translates* vectors from the standard basis to $\mathcal{B}$.

In the example above, left multiplication by

$$M = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

translates vectors from $\mathcal{B}$ to the standard basis.

It turns out that the inverse of M is

$$M^{-1} = \begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{5}{4} \\ \frac{1}{4} & -\frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}.$$

And left multiplication by M^{-1} translates vectors from the standard basis to $\mathcal{B}$.

Example. The "translation" paradigm is a useful one, since it makes it easy to see how to set up arbitrary changes of basis.

For example, suppose

$$\mathcal{B}' = \left\{ \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

is another basis for $\mathbb{R}^3$.

Here's how to translate vectors from $\mathcal{B}'$ to $\mathcal{B}$:

$$(\mathcal{B}' \to \mathcal{B}) = (\text{standard} \to \mathcal{B})(\mathcal{B}' \to \text{standard}).$$

Remember that the product $(\text{standard} \to \mathcal{B})(\mathcal{B}' \to \text{standard})$ is read from right to left! Thus, the composite operation $(\text{standard} \to \mathcal{B})(\mathcal{B}' \to \text{standard})$ translates a $\mathcal{B}'$ vector to a standard vector, and then translates the standard vector to a $\mathcal{B}$ vector. Moreover, I have matrices which perform each of the right-hand operations:

$$\begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix}$$

translates vectors from $\mathcal{B}'$ to the standard basis, and

$$\begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{5}{4} \\ \frac{1}{4} & -\frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix}$$

translates vectors from the standard basis to $\mathcal{B}$.

Therefore, multiplication by

$$\begin{bmatrix} -\frac{1}{4} & \frac{3}{4} & \frac{5}{4} \\ \frac{1}{4} & -\frac{3}{4} & -\frac{1}{4} \\ \frac{1}{4} & \frac{1}{4} & -\frac{1}{4} \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ -1 & 0 & 1 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} \frac{3}{2} & \frac{3}{4} & \frac{7}{4} \\ \frac{1}{2} & \frac{1}{4} & -\frac{3}{4} \\ -\frac{1}{2} & \frac{1}{4} & \frac{1}{4} \end{bmatrix}$$

will translate vectors from $\mathcal{B}'$ to $\mathcal{B}$. $\square$