5-1-1994

Review Problems for the Final - 2.

Example. Find the eigenvalues of

$$\begin{bmatrix} 8 & 6 \\ -12 & -10 \end{bmatrix}.$$

For each eigenvalue, find as many independent eigenvectors as possible.

The characteristic polynomial is $(\lambda - 8)(\lambda + 10) + 72 = \lambda^2 + 2\lambda - 8 = (\lambda + 4)(\lambda - 2)$. The eigenvalues are $\lambda = -4$ and $\lambda = 2$.

For $\lambda = -4$,

$$A + 4I = \begin{bmatrix} 12 & 6 \\ -12 & -6 \end{bmatrix}.$$

The second row is a multiple of the first, and may be ignored. Divide the first row by 6 to get $\langle 2, 1\rangle$. An eigenvector $\langle a, b\rangle$ must satisfy

$$2 \cdot a + 1 \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 1, -2\rangle$.

For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 6 & 6 \\ -12 & -12 \end{bmatrix}.$$

The second row is a multiple of the first, and may be ignored. Divide the first row by 6 to get $\langle 1, 1\rangle$. An eigenvector $\langle a, b\rangle$ must satisfy

$$1 \cdot a + 1 \cdot b = 0.$$

An obvious solution is $\langle a, b\rangle = \langle 1, -1\rangle$. $\square$

Example. Find the inverse of

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 2 & 3 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 1 & -1 & 1 & 0 & 0 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 + r_3}{\to} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 3 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \underset{r_2 \to r_2 - 3r_3}{\to}$$

$$\begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 0 & 1 & -3 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \underset{r_2 \to r_2/2}{\to} \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \underset{r_1 \to r_1 - r_2}{\to}$$

$$\begin{bmatrix} 1 & 0 & 0 & 1 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 1 & 0 & 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The inverse is

$$\begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} \\ 0 & 0 & 1 \end{bmatrix}. \square$$

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3. Find an orthonormal basis for the subspace of $\mathbb{R}^4$ spanned by

$$\{\langle -1,4,1,1\rangle, \langle 1,9,1,2\rangle, \langle -1,8,3,2\rangle\}.$$

$v_1' = \langle -1,4,1,1\rangle$,

$$v_2' = \langle 1,9,1,2\rangle - \frac{\langle 1,9,1,2\rangle \cdot \langle -1,4,1,1\rangle}{\langle -1,4,1,1\rangle \cdot \langle -1,4,1,1\rangle}\langle -1,4,1,1\rangle = \langle 3,1,-1,0\rangle.$$

Finally,

$$v_3' = \langle -1,8,3,2\rangle - \frac{\langle -1,8,3,2\rangle \cdot \langle -1,4,1,1\rangle}{\langle -1,4,1,1\rangle \cdot \langle -1,4,1,1\rangle}\langle -1,4,1,1\rangle -$$
$$\frac{\langle -1,8,3,2\rangle \cdot \langle 3,1,-1,0\rangle}{\langle 3,1,-1,0\rangle \cdot \langle 3,1,-1,0\rangle}\langle 3,1,-1,0\rangle = \frac{1}{11}\langle 5,-2,13,0\rangle.$$

Now just divide each vector by its length:

$$\left\{\frac{1}{\sqrt{19}}\langle -1,4,1,1\rangle, \frac{1}{\sqrt{11}}\langle 3,1,-1,0\rangle, \frac{1}{\sqrt{198}}\langle 5,-2,13,0\rangle\right\}. \square$$

Example. Let $T : V \to V$ be a linear transformation. Prove that the following statements are equivalent:

1. $\ker T \cap \operatorname{im} T = \{0\}$.
2. If $T(Tv) = 0$, then $Tv = 0$.

Suppose $\ker T \cap \operatorname{im} T = \{0\}$. Let $T(Tv) = 0$. Now $Tv \in \ker T$, and visibly $Tv \in \operatorname{im} T$, so $Tv \in \ker T \cap \operatorname{im} T = \{0\}$. Thus, $Tv = 0$.

Conversely, suppose $T(Tv) = 0$ implies $Tv = 0$. Let $w \in \ker T \cap \operatorname{im} T$. I want to show $w = 0$.

$w \in \operatorname{im} T$, so $w = Tv$ for some $v \in V$. $w \in \ker T$, so

$$0 = Tw = T(Tv).$$

By hypothesis, this implies $Tv = 0$. Therefore, $w = 0$. Hence, $\ker T \cap \operatorname{im} T = \{0\}$. $\square$

Example. Give a specific example of a prime ideal that is not maximal.

$\dfrac{\mathbb{Z}[x]}{\langle x\rangle} \approx \mathbb{Z}$, which is a domain, but not a field. Hence, $\langle x\rangle$ is prime, but not maximal. $\square$

Example. Suppose $M \in M_n(\mathbb{R})$, n is an odd number, and $M = -M^T$, where M^T denotes the transpose of M. Prove that $\det M = 0$.

$\det M = \det(-M^T)$. In $-M^T$, each row of M^T is multiplied by -1. By linearity, $\det(-M^T) = (-1)^n \det M^T = -\det M^T$ (since n is odd).

Now

$$\det M = -\det M^T = -\det M, \quad \text{so} \quad 2\det M = 0.$$

Hence, $\det M = 0$. $\square$

Example. Let G be a group of order p^n, where p is prime and $n \geq 1$. Suppose $px = 0$ for all $x \in G$. What is the primary decomposition of G?

$$G \approx \mathbb{Z}_p \times \cdots \times \mathbb{Z}_p,$$

where $\mathbb{Z}_p$ is repeated n times. $\square$

Example. Let $C[0,1]$ be the ring of continuous real-valued functions on $[0,1]$. Addition and multiplication of functions is defined as usual:

$$(f+g)(x) = f(x) + g(x), \quad (fg)(x) = f(x)g(x).$$

Let $I = \left\{f \in C[0,1] \mid f\left(\dfrac{1}{2}\right) = 0\right\}$. Prove that I is a two-sided ideal, and that

$$\frac{C[0,1]}{I} \approx \mathbb{R}.$$

I will do both parts simultaneously. Define $\phi : C[0,1] \to \mathbb{R}$ by

$$\phi(f) = f\left(\frac{1}{2}\right).$$

Observe that

$$\phi(f+g) = (f+g)\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) + g\left(\frac{1}{2}\right) = \phi(f) + \phi(g),$$

$$\phi(fg) = (fg)\left(\frac{1}{2}\right) = f\left(\frac{1}{2}\right) \cdot g\left(\frac{1}{2}\right) = \phi(f)\phi(g).$$

Therefore, ϕ is a ring map.

It is obvious that $\ker \phi = I$. Since the kernel of a ring map is always a two-sided ideal, I is a two-sided ideal.

Finally, if $r \in \mathbb{R}$, let $f(x) = r$ be the constant function mapping $[0,1]$ to r. Then $\phi(f) = f\left(\dfrac{1}{2}\right) = r$. Hence, ϕ is surjective.

By the First Isomorphism Theorem,

$$\frac{C[0,1]}{I} = \frac{C[0,1]}{\ker \phi} \approx \operatorname{im} \phi = \mathbb{R}. \ \square$$

Example. (a) Let V be an inner product space, and let W be a subspace of V. The **orthogonal complement** of W is

$$W^\perp = \{v \in V \mid (v, w) = 0 \text{ for all } w \in W\}.$$

Show that $W^\perp$ is a subspace of V.

$(0, w) = 0$ for all $w \in W$, so $0 \in W^\perp$.
Let $v_1, v_2 \in W^\perp$, $a \in F$, $w \in W$. Then

$$(av_1 + v_2, w) = a(v_1, w) + (v_2, w) = 0 + 0 = 0.$$

Hence, $av_1 + v_2 \in W^\perp$. Therefore, $W^\perp$ is a subspace of V. $\square$

(b) Let W be the subspace of $\mathbb{R}^4$ spanned by $\langle 1,1,0,1\rangle$ and $\langle 0,1,2,-1\rangle$, where $\mathbb{R}^4$ is given the usual inner product. Find a basis for $W^\perp$.

An element $\langle a,b,c,d\rangle \in W^\perp$ must satisfy

$$\langle 1,1,0,1\rangle \cdot \langle a,b,c,d\rangle = 0, \quad \langle 0,1,2,-1\rangle \cdot \langle a,b,c,d\rangle = 0.$$

Thus,

$$\begin{array}{ccccccccc} a & + & b & & & + & d & = & 0 \\ & & b & + & 2c & - & d & = & 0 \end{array}$$

Row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 & 2 \\ 0 & 1 & 2 & -1 \end{bmatrix}$$

This says

$$a = 2c - 2d, \quad b = -2c + d.$$

Therefore,

$$\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} = \begin{bmatrix} 2c - 2d \\ -2c + d \\ c \\ d \end{bmatrix} = c \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \\ 0 \end{bmatrix} + d \cdot \begin{bmatrix} -2 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

$\{\langle 2,-2,1,0\rangle, \langle -2,1,0,1\rangle\}$ is a basis for $W^\perp$. $\square$