Review Session for Test 1.

Example. Prove or disprove:

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & -2 & -1 \\ -3 & 5 & 0 \end{bmatrix}$$

have the same row space.

Row reduce the matrices:

$$\begin{bmatrix} 1 & -1 & 2 \\ -2 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 1 & -2 & -1 \\ -3 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 \\ 0 & 1 & 3 \end{bmatrix}$$

They have the same row reduced echelon form. Hence, they have the same row space. □

Example. Let I be an ideal in a commutative ring R. The **radical** of I is

$$\operatorname{rad} I = \{a \in R \mid a^n \in I, n \in \mathbb{Z}^+\}.$$

Show that $\operatorname{rad} I$ is an ideal.

First, $0 \in I$, so $0 \in \operatorname{rad} I$. In particular, $\operatorname{rad} I \neq \emptyset$.
Next, suppose $x, y \in \operatorname{rad} I$. This means that $x^m, y^n \in I$ for some $m, n \in \mathbb{Z}^+\}$. Then

$$(x+y)^{m+n} = \sum_{i=0}^{m+n} \binom{m+n}{i} x^i y^{m+n-i}.$$

Consider a term $\binom{m+n}{i} x^i y^{m+n-i}$. Either $i \geq m$ or $m + n - i \geq n$.
If $i \geq m$, then $x^i \in I$, so $\binom{m+n}{i} x^i y^{m+n-i} \in I$.
If $m + n - i \geq n$, then $y^{m+n-i} \in I$, so again $\binom{m+n}{i} x^i y^{m+n-i} \in I$.
Therefore, each term of the sum in is I, so the sum is in I. Hence, $(x+y)^{m+n} \in I$, and so $x + y \in \operatorname{rad} I$.
Finally, suppose $a \in R$. Then

$$(ax)^m = a^m x^m \in I,$$

so $ax \in \operatorname{rad} I$. Therefore, $\operatorname{rad} I$ is an ideal. □

Example. Regard $\mathbb{R}$ as a vector space over $\mathbb{Q}$. Show that the set $\{1, \sqrt{2}, \sqrt{3}\}$ is independent.

Let

$$a \cdot 1 + b \cdot \sqrt{2} + c \cdot \sqrt{3} = 0.$$

I need to show that $a = b = c = 0$.
Clearing any denominators, I may assume that $a, b, c \in \mathbb{Z}$. Then

$$a + b\sqrt{2} = -c\sqrt{3},$$

$$(a + b\sqrt{2})^2 = (c\sqrt{3})^2,$$

$$a^2 + 2ab\sqrt{2} + 2b^2 = 3c^2,$$

$$2ab\sqrt{2} = 3c^2 - a^2 - 2b^2,$$
$$8a^2b^2 = (3c^2 - a^2 - 2b^2)^2.$$

Since 2 divides the left side, it divides the right. Since 2 is prime, it follows that $2 \mid 3c^2 - a^2 - 2b^2$, or $3c^2 - a^2 - 2b^2 = 2d$, $d \in \mathbb{Z}$. Then
$$8a^2b^2 = 4d^2,$$
$$2a^2b^2 = d^2.$$

Again, 2 divides d^2, so $2 \mid d$, or $d = 2e$ for $e \in \mathbb{Z}$. So
$$2a^2b^2 = 4e^2,$$
$$(ab)^2 = 2e^2.$$

I can continue this forever, producing an infinite descent. This is a contradiction, since the numbers involved must decrease in absolute value. The only possibility is that the descent terminates when both sides become 0. So
$$a^2b^2 = 0.$$

There are two cases. If $a = 0$,
$$0 = 3c^2 - 2b^2,$$
$$3c^2 = 2b^2,$$
and again by infinite descent, $b = c = 0$.

If on the other hand $b = 0$, then
$$3c^2 - a^2 = 0,$$
$$3c^2 = a^2.$$

Infinite descent gives $a = c = 0$.

Thus, $a = b = c = 0$ in either case. Therefore, $\{1, \sqrt{2}, \sqrt{3}\}$ is independent. □

Example. Let R be a PID. Prove that a nonzero element $x \in R$ is irreducible if and only if $\langle x \rangle$ is maximal.

Suppose x is irreducible. Let $\langle x \rangle \subset I \subset R$. I must show $I = R$ or $I = \langle y \rangle$.

Since R is a PID, $I = \langle y \rangle$ for some y. Now $x \in \langle y \rangle$ so $x = yz$ for some z. Since x is irreducible, either y is a unit or z is a unit. If z is a unit, then $y = z^{-1}x$ shows $y \in \langle x \rangle$, so $\langle x \rangle = \langle y \rangle = I$. If y is a unit, then $I = \langle y \rangle = R$. Hence, $\langle x \rangle$ is maximal.

Now suppose $\langle x \rangle$ is maximal. Suppose $x = yz$. I must show one of the factors is a unit. Now $\langle x \rangle \subset \langle y \rangle$, so by maximality either $\langle x \rangle = \langle y \rangle$ or $\langle y \rangle = R$.

If $\langle x \rangle \subset \langle y \rangle$, then $y = xw$ for some w. Then $x = xwz$. Since $x \neq 0$, I may cancel to obtain $1 = wz$. Hence, z is a unit.

If $\langle y \rangle = R$, then $yw = 1$ for some w, and y is a unit.

Therefore, x is irreducible. □

Example. Prove: In a PID, nonzero prime ideals are maximal.

Let P be a prime ideal in a PID R. Write $P = \langle x \rangle$. Then x is a prime element (proof?), so x is irreducible. Hence, $\langle x \rangle$ is maximal. □

Example. Let $V = \mathbb{R}^3$. Let
$$W = \{\langle x, y, z \rangle \mid x, y, z \in \mathbb{R}, x + 1 = z\}.$$

Prove or disprove: W is a subspace of V.

$\langle 0, 0, 0\rangle \notin W$, so W is not a subspace. $\square$

Example. Is $\langle x^4 + 6\rangle$ a prime ideal in $\mathbb{Z}_5[x]$?

You can check that x^4+6 has no roots in $\mathbb{Z}_5$. However, $x^4+6 = (x^2+2)(x^2+3)$. Now $(x^2+2)(x^2+3) \in \langle x^4+6\rangle$, but by looking at degrees it's clear that neither factor is in the ideal. Hence, the ideal is not prime. $\square$
