2-13-1994

Vector Spaces.

Definition. A **vector space** V over a field F is an abelian group $(V,+)$ together with an operation called **scalar multiplication**, which is a map $F \times V \to V$. If $a \in F$ and $x \in V$, denote the product of a and x under scalar multiplication by ax. Scalar multiplication satisfies the following axioms:

1. If $a, b \in F$ and $x \in V$, then $a(bx) = (ab)x$.
2. If $a, b \in F$ and $x \in V$, then $(a+b)x = ax + bx$.
3. If $a \in F$ and $x, y \in V$, then $a(x+y) = ax + ay$.
4. If $x \in V$, then $1 \cdot x = x$.

The elements of V are called **vectors**; the elements of F are called **scalars**. As usual, the use of words like "multiplication" does not imply that the operations involved look like ordinary "multiplication".

Example. The trivial group $\{0\}$ is a vector space over any field F, using the trivial multiplication. □

Example. If F is a field, F is a vector space over F. The scalar multiplication is just the ordinary multiplication inside F, and the axioms follow trivially from the ring axioms for F. □

Example. Let F be a field, and let F^n denote the product of n copies of F:

$$F^n = \underbrace{F \times \cdots \times F}_{n \text{ times}} = \{(x_1, \ldots, x_n) \mid x_i \in F\}.$$

F^n becomes an abelian group under componentwise addition:

$$(x_1, \ldots, x_n) + (y_1, \ldots, y_n) = (x_1 + y_1, \ldots, x_n + y_n).$$

If $a \in F$ and $(x_1, \ldots, x_n) \in F^n$, define a scalar multiplication by

$$a \cdot (x_1, \ldots, x_n) = (ax_1, \ldots, ax_n).$$

It is easy to verify that F^n becomes an F-vector space with these operations.

This example specializes to many interesting vector spaces.

If $F = \mathbb{R}$, $F^n = \mathbb{R}^n$, n-dimensional Euclidean space. Specifically, $\mathbb{R}^2$ is the usual x-y-plane, and vectors are ordered pairs, as usual:

$$(-1, 0) + (3, 7) = (2, 7), \qquad 4 \cdot (5, \sqrt{3}) = (20, 4\sqrt{3}).$$

If p is prime, $\mathbb{Z}_p$ is a field, and $\mathbb{Z}_p^n$ is a vector space over $\mathbb{Z}_p$. For example,

$$\mathbb{Z}_3 \times \mathbb{Z}_3 = \{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2)\}. \ \square$$

Example. More generally, suppose that V and W are vector spaces over a field F. The **direct sum** of V and W is

$$V \oplus W = \{(v, w) \mid v \in V, w \in W\}.$$

$V \oplus W$ becomes an abelian group under componentwise addition:

$$(v, w) + (v', w') = (v + v', w + w').$$

If $a \in F$ and $(v, w) \in V \oplus W$, define a scalar multiplication by multiplying componentwise:

$$a \cdot (v, w) = (av, aw).$$

It is easy to verify that $V \oplus W$ is an F-vector space with these operations. □

Example. The ring of polynomials with real coefficients $\mathbb{R}[x]$ is a vector space over $\mathbb{R}$. $\mathbb{R}[x]$ is an abelian group under addition of polynomials; scalar multiplication is the same as multiplication by constant polynomials (since $\mathbb{R}$ can be regarded as a subset of $\mathbb{R}[x]$).

With these operations, $\mathbb{R}[x]$ becomes and $\mathbb{R}$-vector space. Unlike some of the other examples, $\mathbb{R}[x]$ is **infinite dimensional** (in a sense to be made more precise shortly). □

Example. Let E and F be fields. If $F \subset E$, F is a **subfield** of E.

E becomes a F-vector space as follows. The addition is the addition operation in E; the scalar multiplication is ordinary multiplication inside E. In other words, if $a \in F$ and $x \in E$, then ax is just the product of a and x as elements of E. The vector space axioms follow from the ring axioms for E. □

Example. Let $C[0, 1]$ denote the continuous real-valued functions defined on the interval $0 \le x \le 1$. Add functions pointwise:

$$(f + g)(x) = f(x) + g(x) \quad \text{for} \quad f, g \in C[0, 1].$$

This makes $C[0, 1]$ into an abelian group; in fact, if functions are multiplied pointwise, it becomes a ring.

If $a \in R$ and $f \in C[0, 1]$, define scalar multiplication in pointwise fashion:

$$(af)(x) = a \cdot f(x).$$

These operations make $C[0, 1]$ into an $\mathbb{R}$-vector space.

Like $\mathbb{R}[x]$, $C[0, 1]$ is infinite dimensional. However, its dimension is uncountable, while $\mathbb{R}[x]$ has countably infinite dimension over $\mathbb{R}$.

You can also define a "dot product" for two vectors $f, g \in C[0, 1]$:

$$f \cdot g = \int_0^1 f(x)g(x)\, dx.$$

(The product of continuous functions is continuous, so $f(x)g(x)$ is certainly Riemann integrable.) □

Example. (**Normed vector spaces**) Let V be a real vector space. A **norm** on V is a function $x \to |x|$ from V to $\mathbb{R}$ satisfying:

1. For all $x \in V$, $|x| \ge 0$, and $|x| = 0$ if and only if $x = 0$.
2. For all $x \in V$ and $r \in \mathbb{R}$, $|rx| = |r||x|$.
3. For all $x, y \in V$, $|x + y| \le |x| + |y|$.

You can use $|\cdot|$ to measure distances: the distance from x to y is just $|x-y|$. This makes a normed linear space into a **metric space**.

With a norm present, you can begin to do analysis. In fact, let $\{x_n\}$ be a sequence of elements of the normed vector space V. $\{x_n\}$ is **Cauchy** if for every $\epsilon > 0$, there is an integer N such that

$$|x_m - x_n| < \epsilon \quad \text{for all} \quad m, n \geq N.$$

A sequence $\{x_n\}$ **converges** to $x \in V$ if if for every $\epsilon > 0$, there is an integer N such that

$$|x - x_n| < \epsilon \quad \text{for all} \quad n \geq N.$$

Finally, a normed vector space is a **complete** if every Cauchy sequence converges. A complete normed vector space is called a **Banach space**. Banach spaces are important objects in analysis. □

The data for a vector space are an abelian group and a field. Two obvious ways to "twiddle" the definition are to replace the abelian group with a ring, and to replace the field with a ring.

$$\left\{\begin{array}{c}\text{a field}\\ \text{acting on}\\ \text{a ring}\end{array}\right\} \leftarrow \left\{\begin{array}{c}\text{a field}\\ \text{acting on}\\ \text{an abelian group}\end{array}\right\} \rightarrow \left\{\begin{array}{c}\text{a ring}\\ \text{acting on}\\ \text{an abelian group}\end{array}\right\}$$

(Obviously, there will be axioms ensuring some kind of "compatibility" among the operations involved.) Objects on the left are called **algebras**; objects on the right are called **modules**.

Elementary operations in vector spaces. In what follows, V will denote a vector space over a field F.

1. $0 \cdot x = 0$ for all $x \in V$.

(Note that the "0" on the left is the additive identity in F; the "0" on the right is the 0-vector in V.)

$$0 \cdot x = (0+0) \cdot x = 0 \cdot x + 0 \cdot x, \quad \text{so} \quad 0 = 0 \cdot x. \ \square$$

2. $(-1) \cdot x = -x$ for all $x \in V$.

(The "-1" on the left is the additive inverse of $1 \in F$; the "$-x$" on the right is the additive inverse of $x \in V$.)

$$(-1) \cdot x + x = (-1) \cdot x + 1 \cdot x = ((-1) + 1) \cdot x = 0 \cdot x = 0.$$

Hence, $(-1) \cdot x$ is the additive inverse of x, which is $-x$. □

Definition. Let V be a vector space over a field F, and let $W \subset V$, $w \neq \emptyset$. W is a **subspace** of V if W is a subgroup of $(V, +)$ and W is closed under scalar multiplication.

Example. If V be a vector space over a field F, $\{0\}$ and V are subspaces of V. □

Example. Let F be a field.

$$\{(x, 0) \mid x \in F\} \subset F \times F$$

is a subspace of $F \times F$. So is $\{(0, y) \mid y \in F\}$. □

Example. Consider the following subsets of the polynomial ring $\mathbb{R}[x]$:

$$V_1 = \{f(x) \in \mathbb{R}[x] \mid f(2) = 0\}, \qquad V_2 = \{f(x) \in \mathbb{R}[x] \mid f(2) = 1\}.$$

V_1 is a subspace; in fact, it's the principal ideal $\langle x - 2\rangle$. V_2 is not a subspace: the constant polynomial $f(x) = 1$ is an element of V_2, but $2 \cdot f(x)$ is not. $\square$

Example. In $\mathbb{R}^2$, the subspaces are the origin, lines through the origin, and the entire plane. In $\mathbb{R}^3$, the subspaces are the origin, lines through the origin, planes through the origin, and the entire space. And so on. $\square$