Math 308
5-5-1994

Solutions to Problem Set 10.

1. List all (isomorphism classes of) abelian groups of order 900. For each group in your list, give the invariant factor decomposition and the primary decomposition. *Be sure your list contains no repetitions!*

$900 = 2^2 \cdot 3^2 \cdot 5^2$; the possible factorizations of the prime powers are:

$$2^2: \quad 2^2, 2 \cdot 2$$
$$3^2: \quad 3^2, 3 \cdot 3$$
$$5^2: \quad 5^2, 5 \cdot 5$$

Primary decomposition	Invariant factor decomposition
$\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$	$\mathbb{Z}_{900}$
$\mathbb{Z}_4 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_5 \times \mathbb{Z}_{180}$
$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$	$\mathbb{Z}_3 \times \mathbb{Z}_{300}$
$\mathbb{Z}_4 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_{15} \times \mathbb{Z}_{60}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_{25}$	$\mathbb{Z}_2 \times \mathbb{Z}_{450}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_9 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_{10} \times \mathbb{Z}_{90}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_{25}$	$\mathbb{Z}_6 \times \mathbb{Z}_{150}$
$\mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_3 \times \mathbb{Z}_3 \times \mathbb{Z}_5 \times \mathbb{Z}_5$	$\mathbb{Z}_{30} \times \mathbb{Z}_{30}$

□

---

2. What is the minimal polynomial of

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}?$$

The characteristic polynomial is $-(\lambda - 1)^3$. Since the matrix is not the identity, it does not satisfy $A - I = 0$. On the other hand,

$$(A - I)^2 = \begin{bmatrix} -1 & -1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, the minimal polynomial is $(x - 1)^2$. □

---

3. Let G be a finite abelian group whose order is not divisible by the square of a prime. Prove that G is cyclic.

The order of G must be

$$|G| = p_1 p_2 \cdots p_n,$$

where the p_i are distinct primes. Hence, the primary decomposition is

$$G \approx \mathbb{Z}_{p_1} \times \mathbb{Z}_{p_2} \times \cdots \mathbb{Z}_{p_n} \approx \mathbb{Z}_{p_1 p_2 \cdots p_n}.$$

Thus, G is cyclic. □

---

4. Let $\mathbb{C}^*$ denote the multiplicative group of nonzero complex numbers. What is the torsion subgroup of $\mathbb{C}^*$?

An element z of order $n > 0$ satisfies $z^n = 1$. Hence, the torsion subgroup consist of the n-th complex roots of unity, for $n = 1, 2, \ldots$. □

---

5. Let H be the subset of $\mathbb{Z} \times \mathbb{Z}$ given by

$$H = \{(a, b) \mid a = b \bmod 4\}.$$

(a) Prove that H is a subgroup of $\mathbb{Z} \times \mathbb{Z}$.

$(0, 0) \in H$, since $0 = 0 \bmod 4$. Suppose $(a, b), (c, d) \in H$, so $a = b \bmod 4$ and $c = d \bmod 4$. Then $a - c = b - d \bmod 4$, so

$$(a, b) - (c, d) = (a - c, b - d) \in H.$$

Hence, H is a subgroup of $\mathbb{Z} \times \mathbb{Z}$. □

(b) If H is not torsion free, find a nontrivial element of H having finite order. If H is torsion free, prove it, and show that H is isomorphic to a direct product of copies of $\mathbb{Z}$.

I'll show that $(0, 0)$ is the only element of finite order in H, so H is torsion free. Suppose $(a, b) \in H$, $n > 0$, and $n \cdot (a, b) = (0, 0)$. Then $na = 0$ and $nb = 0$, so $a = b = 0$. Thus, H is torsion free.

If $(a, b) \in H$, then $a = b \bmod 4$ implies $b = a + 4n$ for some $n \in \mathbb{Z}$. Hence,

$$(a, b) = (a, a + 4n) = a \cdot (1, 1) + n \cdot (0, 4).$$

This shows that $(1, 1)$ and $(0, 4)$ generate H. Thus, H is a finitely generated torsion free abelian group, so it's a product of copies of $\mathbb{Z}$. It can't be generated by a single element, since $(1, 1)$ and $(0, 4)$ are visibly not multiples of any single element. So in fact, it must require two generators, and $H \approx \mathbb{Z} \times \mathbb{Z}$. □

"The best thing for being sad is to learn something." - MERLYN, in T. H. WHITE's *The Once and Future King*