Math 308
2-4-1994

Problem Set 2.

Problem Set 2 is due on Friday, February 4.

1. Let G be an abelian group. A subset $\{x_a\}_{a\in A} \subset G$ is **independent** if

$$b_1 x_{a_1} + \cdots + b_n x_{a_n} = 0, \qquad b_k \in \mathbb{Z}, \quad a_k \in A, \quad n \ge 1$$

implies $b_1 x_{a_1} = \cdots = b_n x_{a_n} = 0$. In words, if a finite linear combination of elements of $\{x_a\}_{a\in A}$ with integer coefficients is 0, then all the terms in the combination are 0.

Note that the set $\{x_a\}_{a\in A}$ need not be finite, or even countable.

For example, the elements $(1,0)$ and $(0,1)$ form an independent set in $\mathbb{Z}\oplus\mathbb{Z}$. On the other hand, the set $\{(2,2),(-1,3),(10,-6)\}$ is not independent in $\mathbb{Z}\oplus\mathbb{Z}$, since

$$3\cdot(2,2) + (-4)\cdot(-1,3) + (-1)\cdot(10,-6) = (0,0).$$

A subset that is not independent is **dependent**.

(a) Show that any subset of $(\mathbb{Q},+)$ containing more than one element but not containing 0 is dependent.

(b) Prove or disprove: The subset $\{(1,1),(1,-1)$ of $\mathbb{Z}\oplus\mathbb{Z}$ is independent.

(c) Let G be an abelian group. An independent subset of G is a **maximal independent set** if it is not properly contained in any other independent subset of G.

Prove that every independent set in an abelian group is contained in a maximal independent sets. [Hint: Let G be an abelian group, X an independent subset of G. Let S denote the set of independent sets which contain X, ordered by inclusion. Use Zorn's lemma to show that S has maximal elements.]

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- In Problem 4, define a map $\phi : \mathbb{Z}_3[x] \to \mathbb{Z}_3[i]$ by $\phi(f) = f(i)$. You may take for granted that ϕ is a ring map. Prove that $\ker\phi = \langle x^2+1\rangle$, then apply the First Isomorphism Theorem.
- In Problem 34, suppose $\phi : m\mathbb{Z} \to n\mathbb{Z}$ is an isomorphism. Begin by proving that $\phi(m) = n$ or $\phi(-m) = n$.

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- In Problem 18, define a surjective ring map $\phi : \mathbb{Q}[x] \to \mathbb{Q}$ having $\langle x\rangle$ as its kernel.

"Every beginning is a consequence — every beginning *ends* something." - Paul Valéry