Math 308
2-27-1994

Solutions to Problem Set 4.

1. Write down your last name. Then let n be the number of letters in your last name. Substitute the value of n in the matrix below and write down the resulting matrix:

$$\begin{bmatrix} 3 & 0 & 7 & -1 & 1 & 4 \\ 2 & 3 & 2 & 0 & 0 & 6 \\ -2 & n & 3 & 1 & 1 & 1 \\ 2 & 3 & 1 & -9 & 0 & 4 \\ 0 & 2 & 0 & 3 & -3 & 5 \end{bmatrix}$$

Finally, use *Mathematica* to row reduce the matrix to row reduced echelon form.

I'll do the row reduction with $n = 7$; your results will probably be different.

$$\begin{bmatrix} 3 & 0 & 7 & -1 & 1 & 4 \\ 2 & 3 & 2 & 0 & 0 & 6 \\ -2 & 7 & 3 & 1 & 1 & 1 \\ 2 & 3 & 1 & -9 & 0 & 4 \\ 0 & 2 & 0 & 3 & -3 & 5 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & \frac{6143}{3293} \\ 0 & 1 & 0 & 0 & 0 & \frac{2636}{3293} \\ 0 & 0 & 1 & 0 & 0 & -\frac{218}{3293} \\ 0 & 0 & 0 & 1 & 0 & \frac{756}{3293} \\ 0 & 0 & 0 & 0 & 1 & -\frac{2975}{3293} \end{bmatrix} \quad \square$$

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2. Write down your last name. Then let n be the number of letters in your last name. Substitute the value of n in the matrix below and write down the resulting matrix:

$$\begin{bmatrix} 3 & 2 & -1 & 0 & 4 & 2 \\ 1 & 1 & n & 2 & -3 & 3 \\ -1 & -2 & 5 & 4 & 0 & -1 \\ 6 & 0 & -3 & 2 & 2 & 2 \end{bmatrix}$$

Then use *Mathematica* to find a basis for the null space of the matrix.

Here is the matrix with $n = 7$:

$$\begin{bmatrix} 3 & 2 & -1 & 0 & 4 & 2 \\ 1 & 1 & 7 & 2 & -3 & 3 \\ -1 & -2 & 5 & 4 & 0 & -1 \\ 6 & 0 & -3 & 2 & 2 & 2 \end{bmatrix}$$

Here is a basis for the null space:

$$\left\{ \begin{bmatrix} -126 \\ -10 \\ -78 \\ 101 \\ 0 \\ 160 \end{bmatrix}, \begin{bmatrix} 14 \\ -30 \\ 14 \\ -29 \\ 8 \\ 0 \end{bmatrix} \right\} \quad \square$$

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3. Let $V = \mathbb{R}^3$, considered as a vector space over $\mathbb{R}$.

(a) Let

$$\mathcal{S} = \left\{ \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \right\}.$$

Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that $\mathcal{S} \subset \mathcal{B}$. Prove that your set $\mathcal{B}$ is a basis.

Put the given vectors into a matrix as its rows and row reduce:

$$\begin{bmatrix} 1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \end{bmatrix}$$

The rows of the row reduced matrix span the same subspace as the two given vectors. So all I need to do is pick a third vector which is not a linear combination of the rows of the row reduced matrix. One such vector is $\langle 0, 0, 1 \rangle$. To check, I throw this vector in with the two given vectors and see if the resulting set is independent.

Suppose

$$a \cdot \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} + b \cdot \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} + c \cdot \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

This is equivalent to the matrix equation

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Solve the system by row reduction. (I can ignore the last column, which will be all zeros no matter what.)

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & -1 & 0 \\ -1 & 1 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Translating this back into equation form, this says $a = b = c = 0$. Thus, the three vectors are independent, so they form a basis for $\mathbb{R}^3$ containing the original two vectors. $\square$

(b) Let

$$\mathcal{T} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} \right\}.$$

Find a basis $\mathcal{B}$ for $\mathbb{R}^3$ such that $\mathcal{B} \subset \mathcal{T}$. Prove that your set $\mathcal{B}$ is a basis.

All I need to do is find three of the four vectors such that the three vectors form an independent set. The first three work. To see this, I'll use the procedure of part (a). It comes down to putting the vectors into a matrix as its columns and row reducing:

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & -2 & 1 \\ 2 & 4 & 1 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

As in part (a), this means the three vectors are independent. Hence, they form a basis for $\mathbb{R}^3$. $\square$

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4. Let V be a vector space and let V_1 and V_2 be subspaces of V. The **sum** of the subspaces is

$$V_1 + V_2 = \{v_1 + v_2 \mid v_1 \in V_1, v_2 \in V_2\}.$$

Show that $V_1 + V_2$ is a subspace of V.

Since $0 \in V_1, V_2$, $0 = 0 + 0 \in V_1 + V_2$. Hence, $V_1 + V_2 \neq \emptyset$.
If $v_1, v_1' \in V_1$ and $v_2, v_2' \in V_2$, then

$$(v_1 + v_2) - (v_1' + v_2') = (v_1 - v_1') + (v_2 - v_2') \in V_1 + V_2,$$

since $v_1 - v_1' \in V_1$ and $v_2 - v_2' \in V_2$. Therefore, $V_1 + V_2$ is an additive subgroup of $(V, +)$.
Finally, let $a \in F$, $v_1 \in V_1$, $v_2 \in V_2$. Then

$$a(v_1 + v_2) = av_1 + av_2 \in V_1 + V_2,$$

since $av_1 \in V_1$ and $av_2 \in V_2$.
Therefore, $V_1 + V_2$ is a subspace of V. □

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5. Let V be the vector space of functions $\mathbb{R} \to \mathbb{R}$.

(a) Let

$$V_e = \{f \in V \mid f(-x) = f(x) \text{ for all } x \in \mathbb{R}\},$$
$$V_o = \{f \in V \mid f(-x) = -f(x) \text{ for all } x \in \mathbb{R}\}.$$

(V_e is the set of **even functions** and V_o is the set of **odd functions**.) Show that V_e and V_o are subspaces of V.

First, the constant function 0 is both even and odd, so V_e and V_o are nonempty.
Let $f, g \in V_e$, and let $a \in \mathbb{R}$. Then

$$(af + g)(-x) = af(-x) + g(-x) = af(x) + g(x) = (af + g)(x).$$

Therefore, $af + g \in V_e$, and V_e is a subspace.
Let $f, g \in V_o$, and let $a \in \mathbb{R}$. Then

$$(af + g)(-x) = af(-x) + g(-x) = -af(x) - g(x) = -(af(x) + g(x)) = -(af + g)(x).$$

Therefore, $af + g \in V_o$, and V_o is a subspace. □

(b) Show that $V_e + V_o = V$, but $V_e \cap V_o = \{0\}$.

Let $f \in V$. Then

$$f(x) = \frac{1}{2}(f(x) + f(-x)) + \frac{1}{2}(f(x) - f(-x)).$$

Now

$$\frac{1}{2}(f(-x) + f(-(-x))) = \frac{1}{2}(f(-x) + f(x)) = \frac{1}{2}(f(x) + f(-x)),$$
$$\frac{1}{2}(f(-x) - f(-(-x))) = \frac{1}{2}(f(-x) - f(x)) = -\frac{1}{2}(f(x) - f(-x)).$$

This proves that $\frac{1}{2}(f(x) + f(-x))$ is even and $\frac{1}{2}(f(x) - f(-x))$ is odd. Therefore, $V = V_e + V_o$.

On the other hand, suppose $f \in V_e \cap V_o$. Then

$$f(x) = f(-x) = -f(x),$$

so $2f(x) = 0$ and $f(x) = 0$. Therefore, $V_e \cap V_o = \{0\}$. $\square$

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5. Determine whether the set $\{\langle 2,-1,0\rangle, \langle 1,2,5\rangle, \langle 7,-1,5\rangle\}$ is linearly independent over $\mathbb{R}$.

Suppose

$$a \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + b \cdot \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + c \cdot \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

You can solve for a, b, and c directly, using basic algebra. I'll use row reduction.
The system is equivalent to the matrix equation

$$\begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 5 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

To solve it, row reduce the coefficient matrix. (I don't need to include the all zero column, because it won't change.)

$$\begin{bmatrix} 2 & 1 & 7 \\ -1 & 2 & -1 \\ 0 & 5 & 5 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

This is equivalent to the equations

$$a + 3c = 0, \quad b + c = 0.$$

If I set $c = 1$ (for instance), I get $a = -3$, $b = -1$. Thus,

$$(-3) \cdot \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} + (-1) \cdot \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix} + 1 \cdot \begin{bmatrix} 7 \\ -1 \\ 5 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since a nontrivial linear combination of the vectors equals $\vec{0}$, the set is dependent. $\square$

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7. If $\{u, v, w\}$ is a linearly independent subset of a vector space, show that $\{u, u+v, u+v+w\}$ is also linearly independent.

Suppose

$$au + b(u+v) + c(u+v+w) = 0.$$

This means that

$$(a+b+c)u + (b+c)v + cw = 0.$$

Since $\{u, v, w\}$ is linearly independent,

$$a+b+c = 0, \quad b+c = 0, \quad c = 0.$$

Substituting the last equation into the second yields $b = 0$. Then plugging $c = 0$ and $b = 0$ into the first equation gives $a = 0$. Hence, the set $\{u, u+v, u+v+w\}$ is also linearly independent. $\square$

8. If $\{v_1, v_2, \ldots, v_n\}$ is a linearly dependent set of vectors, prove that one of these vectors is a linear combination of the others.

Since the set is linearly dependent, there are numbers $a_1, \ldots, a_n$, not all 0, such that

$$a_1 v_1 + \cdots + a_n v_n = 0.$$

Suppose that $a_k \neq 0$. Then

$$v_k = -a_k^{-1}\left(a_2 v_2 + \cdots + \widehat{a_k v_k} + \cdots + a_n v_n\right).$$

(The $\widehat{\quad}$ means that the term is omitted.) Thus, v_k is a linear combination of the other vectors. □

11. If V is a vector space over F of dimension 5 and U and W are subspaces of V of dimension 3, prove that $U \cap W \neq \{0\}$. Generalize.

Suppose $\{u_1, u_2, u_3\}$ is a basis for U and $\{w_1, w_2, w_3\}$ is a basis for W. Since V has dimension 5, the set $\{u_1, u_2, u_3, w_1, w_2, w_3\}$ is dependent. Hence, there are numbers a_1, a_2, a_3, b_1, b_2, b_3, not all 0, such that

$$a_1 u_1 + a_2 u_2 + a_3 u_3 + b_1 w_1 + b_2 w_2 + b_3 w_3 = 0.$$

If all the b's are 0, then at least one of the a's is nonzero, and

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = 0.$$

This is a nontrivial linear combination of the u's adding up to 0, contradicting independence of the u's. If all the a's are 0, then at least one of the b's is nonzero, and

$$b_1 w_1 + b_2 w_2 + b_3 w_3 = 0.$$

This is a nontrivial linear combination of the w's adding up to 0, contradicting independence of the w's. Now

$$a_1 u_1 + a_2 u_2 + a_3 u_3 = -b_1 w_1 - b_2 w_2 - b_3 w_3,$$

and the preceding discussion shows that at least one of the a's is nonzero and at least one of the b's is nonzero. Therefore, this expression is a nonzero vector which lies in both U and W, so $U \cap W \neq \{0\}$.

In general, if U and W are subspaces of V such that $\dim U + \dim W > \dim V$, then $U \cap W \neq \{0\}$. □

15. Let $V = \mathbb{R} \oplus \mathbb{R} \oplus \mathbb{R}$ and $W = \{(a, b, c) \mid a + b = c\}$. Is W a subspace of V? If so, what is its dimension?

$(0, 0, 0) \in W$, so W is nonempty.

Suppose $(a, b, c), (a', b', c') \in W$, so $a + b = c$ and $a' + b' = c'$. Then

$$(a, b, c) - (a', b', c') = (a - a', b - b', c - c').$$

Since $(a - a') + (b - b') = c - c'$, $(a - a', b - b', c - c') \in W$. Therefore, W is an additive subgroup.

Now suppose $k \in F$. Then

$$k \cdot (a, b, c) = (ka, kb, kc).$$

Since $ka + kb = kc$, $(ka, kb, kc) \in W$. Hence, W is a subspace.

W has dimension 2; in fact, $\{\langle 1, 0, 1\rangle, \langle 0, 1, 1\rangle\}$ is a basis. The vectors are elements of W by inspection. They are independent, because

$$a \cdot \langle 1, 0, 1\rangle + b \cdot \langle 0, 1, 1\rangle = \langle 0, 0, 0\rangle$$

implies $a = 0$ and $b = 0$ (by examining the first and second components).

Finally, if $\langle a, b, c\rangle \in W$, then $c = a + b$ implies

$$\langle a, b, c\rangle = \langle a, b, a + b\rangle = a \cdot \langle 1, 0, 1\rangle + b \cdot \langle 0, 1, 1\rangle.$$

Hence, the two vectors span W. $\square$

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25. Let $u = \langle 2, 3, 1\rangle$, $v = \langle 1, 3, 0\rangle$, $w = \langle 2, -3, 3\rangle$. Since $\frac{1}{2}u - \frac{2}{3}v - \frac{1}{6}w = \langle 0, 0, 0\rangle$, can we conclude that u, v, and w are linearly dependent over $\mathbb{Z}_7$?

$2 \cdot 4 = 1 \bmod 7$, $3 \cdot 5 = 1 \bmod 7$, and $6 \cdot 6 = 1 \bmod 7$. Thus, $\frac{1}{2} = 4$, $-\frac{2}{3} = (-2) \cdot 5 = -10 = 4$, and $\frac{1}{6} = 6 \bmod 7$. The equation is

$$4 \cdot \langle 2, 3, 1\rangle + 4 \cdot \langle 1, 3, 0\rangle + 6 \cdot \langle 2, -3, 3\rangle = \langle 0, 0, 0\rangle.$$

This is a nontrivial linear combination of the vectors adding up to $\vec{0}$, so the vectors are dependent. $\square$

"When the gods wish to punish us they answer our prayers." - Oscar Wilde

