Math 308
3-15-1994

Solutions to Problem Set 5.

1. Row reduce the following matrix to row reduced echelon form *by hand*. You should say what operations you're performing at each step, particularly if you want to be lazy and perform several operations at a time. You may (and should) check your answer using *Mathematica*, but you won't get credit if you don't show your work.

$$\begin{bmatrix} 3 & 1 & -9 & -1 & 0 & 3 \\ 1 & 0 & -2 & 0 & 1 & 1 \\ -6 & -1 & 15 & 2 & -1 & -5 \end{bmatrix}$$

$$\begin{bmatrix} 3 & 1 & -9 & -1 & 0 & 3 \\ 1 & 0 & -2 & 0 & 1 & 1 \\ -6 & -1 & 15 & 2 & -1 & -5 \end{bmatrix} \xrightarrow[r_1 \leftrightarrow r_2]{} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 1 \\ 3 & 1 & -9 & -1 & 0 & 3 \\ -6 & -1 & 15 & 2 & -1 & -5 \end{bmatrix} \xrightarrow[r_2 \to r_2 - 3r_1]{}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 & -3 & 0 \\ -6 & -1 & 15 & 2 & -1 & -5 \end{bmatrix} \xrightarrow[r_3 \to r_3 + 6r_1]{} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 & -3 & 0 \\ 0 & -1 & 3 & 2 & 5 & 1 \end{bmatrix} \xrightarrow[r_3 \to r_3 + r_2]{}$$

$$\begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & -3 & -1 & -3 & 0 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \xrightarrow[r_2 \to r_2 + r_3]{} \begin{bmatrix} 1 & 0 & -2 & 0 & 1 & 1 \\ 0 & 1 & -3 & 0 & -1 & 1 \\ 0 & 0 & 0 & 1 & 2 & 1 \end{bmatrix} \quad \square$$

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2. Solve the following system of equations over $\mathbb{Z}_5$:

$$\begin{array}{rcrcrcrcl} 3w & + & 2x & + & 4y & + & z & = & 1 \\ 2w & + & 3x & + & & & 2z & = & 1 \\ w & + & 3x & + & 4y & + & 2z & = & 3 \end{array}$$

You should write your solution in terms of the elements of $\mathbb{Z}_5$, i.e. $\{0, 1, 2, 3, 4\}$ — no fractions or negative numbers.

Row reduce the corresponding matrix:

$$\begin{bmatrix} 3 & 2 & 4 & 1 & 1 \\ 2 & 3 & 0 & 2 & 1 \\ 1 & 3 & 4 & 2 & 3 \end{bmatrix} \quad \to \quad \begin{bmatrix} 1 & 0 & 0 & \frac{39}{8} & \frac{35}{8} \\ 0 & 1 & 0 & \frac{3}{4} & \frac{3}{4} \\ 0 & 0 & 1 & \frac{159}{32} & \frac{11}{32} \end{bmatrix}$$

Now $2 \cdot 8 = 1 \bmod 5$, so $2 = 8^{-1}$. $4 \cdot 4 = 1 \bmod 5$, so $4 = 4^{-1}$. $3 \cdot 32 = 1 \bmod 5$, so $3 = 32^{-1}$. This allows me to rewrite the fractions in the last matrix:

$$\frac{39}{8} = 2 \cdot 39 = 3, \quad \frac{35}{8} = 2 \cdot 35 = 0, \quad \frac{3}{4} = 4 \cdot 3 = 2, \quad \frac{159}{32} = 3 \cdot 159 = 2, \quad \frac{11}{32} = 3 \cdot 11 = 3.$$

The matrix becomes

$$\begin{bmatrix} 1 & 0 & 0 & 3 & 0 \\ 0 & 1 & 0 & 2 & 2 \\ 0 & 0 & 1 & 2 & 3 \end{bmatrix}$$

The corresponding equations are

$$w + 3z = 0, \quad x + 2z = 2, \quad y + 2z = 3.$$

The parametrized solution is

$$w = -3z = 2z, \quad x = -2z + 2 = 3z + 2, \quad y = -2z + 3 = 3z + 3. \square$$

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3. Replace n in the following matrix with the number of letters in your last name:

$$\begin{bmatrix} -2 & 4 & 5 & 0 & 2 & 1 \\ 1 & 3 & 0 & -4 & 6 & 7 \\ 1 & 2 & n & 2 & 4 & 5 \end{bmatrix}$$

Then find bases for the row space and the column space of the matrix.

I'll do it with $n = 7$. Obviously, your mileage may differ.

$$\begin{bmatrix} -2 & 4 & 5 & 0 & 2 & 1 \\ 1 & 3 & 0 & -4 & 6 & 7 \\ 1 & 2 & 7 & 2 & 4 & 5 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 0 & -\frac{14}{25} & \frac{42}{25} & \frac{12}{5} \\ 0 & 1 & 0 & -\frac{86}{75} & \frac{36}{25} & \frac{23}{15} \\ 0 & 0 & 1 & \frac{52}{75} & -\frac{2}{25} & -\frac{1}{15} \end{bmatrix}$$

The vectors

$$\left\langle 1, 0, 0, -\frac{14}{25}, \frac{42}{25}, \frac{12}{5} \right\rangle, \left\langle 0, 1, 0, -\frac{86}{75}, \frac{36}{25}, \frac{23}{15} \right\rangle, \left\langle 0, 0, 1, \frac{52}{75}, -\frac{2}{25}, -\frac{1}{15} \right\rangle$$

form a basis for the row space.

The leading coefficients occur in columns 1, 2, and 3. Hence, the vectors

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 5 \\ 0 \\ 7 \end{bmatrix}$$

form a basis for the column space. $\square$

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4. Determine whether the vector $\langle -5, -16, 10 \rangle$ is in the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\langle 1, -4, 2 \rangle, \langle 2, 1, -1 \rangle, \text{ and } \langle 4, 11, -7 \rangle.$$

You should either prove it isn't, or write it as a linear combination of the three vectors.

I need to find numbers a, b, and c such that

$$a \cdot \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} + b \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + c \cdot \begin{bmatrix} 4 \\ 11 \\ -7 \end{bmatrix} = \begin{bmatrix} -5 \\ -16 \\ 10 \end{bmatrix}.$$

Row reduce the corresponding matrix:

$$\begin{bmatrix} 1 & 2 & 4 & -5 \\ -4 & 1 & 11 & -16 \\ 2 & -1 & -7 & 10 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & -2 & 3 \\ 0 & 1 & 3 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

The last matrix says:

$$a - 2c = 3, \quad b + 3c = -4.$$

Set $c = 0$ (for instance). Then $a = 3$, $b = -4$. Thus,

$$3 \cdot \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} + (-4) \cdot \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + 0 \cdot \begin{bmatrix} 4 \\ 11 \\ -7 \end{bmatrix} = \begin{bmatrix} -5 \\ -16 \\ 10 \end{bmatrix}. \square$$

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5. Let V be a vector space, and let V_1 and V_2 be subspaces of V. Suppose that $V = V_1 + V_2$, and $V_1 \cap V_2 = \{0\}$. Show that every vector $v \in V$ can be written *uniquely* as $v = x + y$, where $x \in V_1$ and $y \in V_2$.

Since $V = V_1 + V_2$, every vector $v \in V$ can be written as $v = x + y$, where $x \in V_1$ and $y \in V_2$. I have to show this expression is unique.

Suppose that

$$x + y = x' + y', \quad \text{where} \quad x, x' \in V_1, y, y' \in V_2.$$

Then $x - x' = y' - y$. This is an element of $V_1 \cap V_2$, and $V_1 \cap V_2 = \{0\}$. Therefore, $x - x' = 0 = y' - y$, i.e. $x = x'$ and $y = y'$.

This proves that every vector $v \in V$ can be written *uniquely* as $v = x + y$, where $x \in V_1$ and $y \in V_2$. $\square$

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6. Let

$$S = \left\{ \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 3 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} \right\}.$$

Either find a subset of S which is a basis for $\mathbb{R}^4$, or show that no subset of S is a basis for $\mathbb{R}^4$.

Put the vectors into a matrix as the *columns* and row reduce:

$$\begin{bmatrix} -3 & 1 & -2 & 3 & 2 \\ 2 & 1 & 3 & -2 & 1 \\ 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 1 & 7 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

The leading coefficients in the row reduced echelon matrix occur in columns 1, 2, 4, and 5. Therefore, the vectors

$$T = \left\{ \begin{bmatrix} -3 \\ 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -2 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 7 \end{bmatrix} \right\}$$

form a basis for the subspace spanned by S. Since there are four vectors in T, the subspace has dimension 4. Therefore, the subspace spanned by S is all of $\mathbb{R}^4$, and T is a basis for $\mathbb{R}^4$. $\square$

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7. Prove or disprove: If $\{x, y, z\}$ is an independent set of vectors in a real vector space V, then the set $\{x - y, y + z, x + 2z\}$ is independent.

Suppose

$$a(x - y) + b(y + z) + c(x + 2z) = 0.$$

Then

$$(a + c)x + (-a + b)y + (b + 2c)z = 0.$$

Since $\{x, y, z\}$ is independent,

$$a + c = 0, \quad -a + b = 0, \quad b + 2c = 0.$$

From the second equation, $a = b$; plugging this into the first equation, $b + c = 0$. Subtracting this from the third equation yields $c = 0$. Setting $c = 0$ in $b + c = 0$ gives $b = 0$; then $a = b$ implies $a = 0$.

Hence, the set $\{x - y, y + z, x + 2z\}$ is independent. □

"Some persons are very decisive when it comes to avoiding decisions." - Brendan Francis

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