3-23-1994

Problem Set 7.

Problem Set 7 is due Wednesday, March 30.

1. (a) Prove: If $f : V \to W$ is an injective linear transformation and $\{v_i\}_I$ is independent in V, then $\{f(v_i)\}_I$ is independent in W.

(b) Prove: If $f : V \to W$ is a surjective linear transformation and $\{v_i\}_I$ spans V, then $\{f(v_i)\}_I$ spans W.

The following is an immediate consequence of (a) and (b).

Corollary. If $f : V \to W$ is an isomorphism of vector spaces and $\{v_i\}_I$ is a basis for V, then $\{f(v_i)\}_I$ is a basis for W. $\square$

(c) Prove: If $f : V \to W$ is an injective linear transformation and $\dim V = \dim W < \infty$, then f is an isomorphism.

(d) If $f : V \to V$ is an injective linear transformation, show by explicit counterexample that f need not be surjective if V is not finite dimensional. (Hint: Look at $\mathbb{R}[x]$ as an $\mathbb{R}$ vector space.)

(e) Let $f : V \to W$ be a linear transformation, and let $\{v_i\}_I$ be a basis for V. If $\{f(v_i)\}_I$ is independent in W, then f is injective.

(f) Prove: If $f : V \to W$ is a surjective linear transformation and $\dim V = \dim W < \infty$, then f is an isomorphism.

2. The solution space of

$$x - 2y + 4z = 0$$

is a subspace of $\mathbb{R}^3$ (a plane, in fact).

(a) The vector $v_1 = \langle 1, -2, 4\rangle$ is perpendicular to the plane. (Why?) The vector $v_2 = \langle 2, 1, 0\rangle$ lies in the plane. (Why?) Find a nonzero vector v_3 which lies in the plane and is perpendicular to v_2. (You may use the fact that two vectors are perpendicular if and only if their dot product is 0.)

(b) $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$. Therefore, if $v \in \mathbb{R}^3$, there is a unique way to express it as a linear combination of v_1, v_2, and v_3:

$$v = av_1 + bv_2 + cv_3.$$

Define a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ by

$$\pi(v) = bv_2 + cv_3.$$

Compute $[\pi]_{\mathcal{B},\mathcal{B}}$, the matrix of π relative to $\mathcal{B}$ (on the domain and the range). Use this to compute $[\pi]_{\mathrm{std},\mathrm{std}}$, the matrix of π relative to the standard basis for $\mathbb{R}^3$ (on the domain and the range).

3. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(a) Write down the matrix $[\mathcal{B} \to \mathrm{std}]$; i.e. the matrix which takes a vector written in terms of $\mathcal{B}$ and translates it into a standard vector:

$$[\mathcal{B} \to \mathrm{std}] \cdot v_{\mathcal{B}} = v_{\mathrm{std}}.$$

(b) Find the matrix $[\mathrm{std} \to \mathcal{B}]$ which takes a vector written in terms of the standard basis and translates it into a $\mathcal{B}$ vector:

$$[\mathrm{std} \to \mathcal{B}] \cdot v_{\mathrm{std}} = v_{\mathcal{B}}.$$

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix relative to the standard bases on the domain and range is

$$[T]_{\text{std},\text{std}} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

Find $T_{\mathcal{B},\mathcal{B}}$.

4. Let $P_2[x]$ denote the real vector space of polynomials of degree less than or equal to 2 with real coefficients. $P_2[x]$ becomes an inner product space with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\, dx.$$

(a) The set

$$\mathcal{B} = \left\{1, \sqrt{12}\left(x - \frac{1}{2}\right), 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)\right\}$$

is an orthonormal basis for $P_2[x]$. Check part of this assertion by showing that $\sqrt{12}\left(x - \dfrac{1}{2}\right)$ has length 1, and that 1 and $6\sqrt{5}\left(x^2 - x + \dfrac{1}{6}\right)$ are perpendicular. (You should carry out these computations by hand.)

(b) Find the components of $x^2 - 2x + 2$ with respect to the basis $\mathcal{B}$. (The computations are extremely messy; you should carry them out using *Mathematica*.)

"If a friend telleth thee a fault, imagine always that he telleth thee not the whole." - THOMAS FULLER