Math 308
5-3-1994

Solutions to Problem Set 7.

1. (a) Prove: If $f : V \to W$ is an injective linear transformation and $\{v_i\}_I$ is independent in V, then $\{f(v_i)\}_I$ is independent in W.

Suppose

$$a_1 f(v_{i_1}) + \cdots + a_n f(v_{i_n}) = 0, \qquad a_i \in F.$$

f is a linear transformation, so

$$f\left(a_1 v_{i_1} + \cdots + a_n v_{i_n}\right) = 0.$$

f is injective, so

$$a_1 v_{i_1} + \cdots + a_n v_{i_n} = 0.$$

$\{v_i\}_I$ is independent, so $a_1 = \cdots = a_n = 0$. Therefore, $\{f(v_i)\}_I$ is independent in W. □

(b) Prove: If $f : V \to W$ is a surjective linear transformation and $\{v_i\}_I$ spans V, then $\{f(v_i)\}_I$ spans W.

Let $w \in W$. Since f is surjective, I may find $v \in V$ such that $f(v) = w$. Since $\{v_i\}_I$ spans V, there are scalars $a_1, \ldots, a_n$ such that

$$v = a_1 v_{i_1} + \cdots + a_n v_{i_n}.$$

Then

$$w = f(v) = f\left(a_1 v_{i_1} + \cdots + a_n v_{i_n}\right) = a_1 f(v_{i_1}) + \cdots + a_n f(v_{i_n}).$$

Therefore, $\{f(v_i)\}_I$ spans W. □

The following is an immediate consequence of (a) and (b).

Corollary. If $f : V \to W$ is an isomorphism of vector spaces and $\{v_i\}_I$ is a basis for V, then $\{f(v_i)\}_I$ is a basis for W. □

(c) Prove: If $f : V \to W$ is an injective linear transformation and $\dim V = \dim W < \infty$, then f is an isomorphism.

It suffices to show that f is surjective. Let $v_1, \ldots, v_n$ be a basis for V. By (a), $f(v_1), \ldots, f(v_n)$ is independent. Every independent set can be enlarged to a basis, but this set already contains $n = \dim V = \dim W$ vectors. Therefore, it *is* a basis.

Hence, if $w \in W$, there are scalars $a_1, \ldots, a_n$ such that

$$w = a_1 f(v_1) + \cdots a_n f(v_n).$$

So

$$w = f\left(a_1 v_1 + \cdots + a_n v_n\right).$$

Therefore, f is surjective, so it's an isomorphism. □

(d) If $f : V \to V$ is an injective linear transformation, show by explicit counterexample that f need not be surjective if V is not finite dimensional.

Define a linear transformation $\phi : \mathbb{R}[x] \to \mathbb{R}[x]$ by

$$\phi\left[f(x)\right] = x \cdot f(x).$$

If $f, g \in \mathbb{R}[x]$ and $a \in \mathbb{R}$,

$$\phi\left[af(x) + g(x)\right] = x \cdot \left[af(x) + g(x)\right] = ax \cdot f(x) + x \cdot g(x) = a\phi\left[f(x)\right] + \phi\left[g(x)\right].$$

Hence, ϕ is a linear transformation. If $\phi\left[f(x)\right] = 0$, then $x \cdot f(x) = 0$, and this is only true if $f(x) = 0$. Hence, $\ker\phi = \{0\}$, so ϕ is injective. However, ϕ is not surjective, because there is no $f(x) \in \mathbb{R}[x]$ such that $\phi\left[f(x)\right] = 1$. □

(e) Let $f : V \to W$ be a linear transformation, and let $\{v_i\}_I$ be a basis for V. If $\{f(v_i)\}_I$ is independent in W, then f is injective.

I will show $\ker f = \{0\}$. Suppose $v \in V$, $f(v) = 0$. Write

$$v = a_1 v_{i_1} + \cdots + a_n v_{i_n}, \qquad a_i \in F.$$

Then

$$0 = f(v) = f\left(a_1 v_{i_1} + \cdots + a_n v_{i_n}\right) = a_1 f\left(v_{i_1}\right) + \cdots + a_n f\left(v_{i_n}\right).$$

Since $\{f(v_i)\}_I$ is independent, $a_1 = \cdots = a_n = 0$. Then

$$v = a_1 v_{i_1} + \cdots + a_n v_{i_n} = 0.$$

Therefore, $\ker f = \{0\}$, so f is injective. □

(f) Prove: If $f : V \to W$ is a surjective linear transformation and $\dim V = \dim W < \infty$, then f is an isomorphism.

Let $v_1, \ldots, v_n$ be a basis for V, where $n = \dim V$. Then $v_1, \ldots, v_n$ spans V, so by (b) $f(v_1), \ldots, f(v_n)$ spans W. Some subset of a spanning set is a basis, but this spanning set already has $n = \dim V = \dim W$ elements. Hence, it *is* a basis. In particular, $f(v_1), \ldots, f(v_n)$ is independent, so by (e) f is injective. Hence, it's an isomorphism. □

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2. The solution space of

$$x - 2y + 4z = 0$$

is a subspace of $\mathbb{R}^3$ (a plane, in fact).

(a) The vector $v_1 = \langle 1, -2, 4\rangle$ is perpendicular to the plane. (Why?) The vector $v_2 = \langle 2, 1, 0\rangle$ lies in the plane. (Why?) Find a nonzero vector v_3 which lies in the plane and is perpendicular to v_2. (You may use the fact that two vectors are perpendicular if and only if their dot product is 0.)

A normal vector to the plane can be constructed from the coefficients of x, y, and z: $\langle 1, -2, 4\rangle$. This is v_1. Since the plane consists of everything perpendicular to v_1, and since $v_1 \cdot v_2 = 0$, v_2 lies in the plane. Finally, I want $v_3 = \langle a, b, c\rangle$ such that

$$\langle a, b, c\rangle \cdot \langle 1, -2, 4\rangle = 0 \quad \text{and} \quad \langle a, b, c\rangle \cdot \langle 2, 1, 0\rangle = 0.$$

That is,

$$a - 2b + 4c = 0 \quad \text{and} \quad 2a + b = 0.$$

Row reduce the coefficient matrix:

$$\begin{bmatrix} 1 & -2 & 4 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{4}{5} \\ 0 & 1 & -\frac{8}{5} \end{bmatrix}$$

This says $a = -\frac{4}{5}c$, $b = \frac{8}{5}c$. Set $c = 5$; I get $a = -4$, $b = 8$. $v_3 = \langle -4, 8, 5 \rangle$ is one possible solution. □

(b) $\mathcal{B} = \{v_1, v_2, v_3\}$ is a basis for $\mathbb{R}^3$. Therefore, if $v \in \mathbb{R}^3$, there is a unique way to express it as a linear combination of v_1, v_2, and v_3:

$$v = av_1 + bv_2 + cv_3.$$

Define a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^3$ by

$$\pi(v) = bv_2 + cv_3.$$

Compute $[\pi]_{\mathcal{B},\mathcal{B}}$, the matrix of π relative to $\mathcal{B}$ (on the domain and the range). Use this to compute $[\pi]_{\text{std,std}}$, the matrix of π relative to the standard basis for $\mathbb{R}^3$ (on the domain and the range).

Since $\pi(v_1) = 0$, $\pi(v_2) = v_2$, and $\pi(v_3) = v_3$,

$$[\pi]_{\mathcal{B},\mathcal{B}} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

$$[\mathcal{B} \to \text{std}] = \begin{bmatrix} 1 & 2 & -4 \\ -2 & 1 & 8 \\ 4 & 0 & 5 \end{bmatrix}.$$

Taking the inverse, I find that

$$[\text{std} \to \mathcal{B}] = \frac{1}{105} \begin{bmatrix} 5 & -10 & 20 \\ 42 & 21 & 0 \\ -4 & 8 & 5 \end{bmatrix}.$$

So

$$[\pi]_{\text{std,std}} = \begin{bmatrix} 1 & 2 & -4 \\ -2 & 1 & 8 \\ 4 & 0 & 5 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \frac{1}{105} \begin{bmatrix} 5 & -10 & 20 \\ 42 & 21 & 0 \\ -4 & 8 & 5 \end{bmatrix} = \frac{1}{21} \begin{bmatrix} 20 & 2 & -4 \\ 2 & 17 & 8 \\ -4 & 8 & 5 \end{bmatrix}. \square$$

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3. Let

$$\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}.$$

(a) Write down the matrix $[\mathcal{B} \to \text{std}]$; i.e. the matrix which takes a vector written in terms of $\mathcal{B}$ and translates it into a standard vector:

$$[\mathcal{B} \to \text{std}] \cdot v_{\mathcal{B}} = v_{\text{std}}.$$

$$[\mathcal{B} \to \text{std}] = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}. \square$$

(b) Find the matrix $[\text{std} \to \mathcal{B}]$ which takes a vector written in terms of the standard basis and translates it into a $\mathcal{B}$ vector:

$$[\text{std} \to \mathcal{B}] \cdot v_{\text{std}} = v_{\mathcal{B}}.$$

$[\text{std} \to \mathcal{B}]$ is simply the inverse of $[\mathcal{B} \to \text{std}]$:

$$[\text{std} \to \mathcal{B}] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}. \square$$

(c) Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation whose matrix relative to the standard bases on the domain and range is

$$[T]_{\text{std,std}} = \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 4 & 0 \end{bmatrix}.$$

Find $T_{\mathcal{B},\mathcal{B}}$.

$$T_{\mathcal{B},\mathcal{B}} = [\text{std} \to \mathcal{B}][T]_{\text{std,std}}[\mathcal{B} \to \text{std}] = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 2 & 1 & 1 \\ 1 & 4 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & -1 \\ -2 & -\frac{5}{2} & -\frac{1}{2} \\ 7 & \frac{1}{2} & \frac{9}{2} \end{bmatrix}. \square$$

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4. Let $P_2[x]$ denote the real vector space of polynomials of degree less than or equal to 2 with real coefficients. $P_2[x]$ becomes an inner product space with the inner product

$$\langle f, g \rangle = \int_0^1 f(x)g(x)\,dx.$$

(a) The set

$$\mathcal{B} = \left\{ 1, \sqrt{12}\left(x - \frac{1}{2}\right), 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right) \right\}$$

is an orthonormal basis for $P_2[x]$. Check part of this assertion by showing that $\sqrt{12}\left(x - \frac{1}{2}\right)$ has length 1, and that 1 and $6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right)$ are perpendicular.

$$\int_0^1 \left[\sqrt{12}\left(x - \frac{1}{2}\right)\right]^2 dx = 12\int_0^1 \left(x^2 - x + \frac{1}{4}\right) dx = 12\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{4}x\right]_0^1 = 1.$$

$$\int_0^1 1 \cdot 6\sqrt{5}\left(x^2 - x + \frac{1}{6}\right) dx = 6\sqrt{5}\left[\frac{1}{3}x^3 - \frac{1}{2}x^2 + \frac{1}{6}x\right]_0^1 = 0. \square$$

(b) Find the components of $x^2 - 2x + 2$ with respect to the basis $\mathcal{B}$.

Using *Mathematica*, I find that

$$x^2 - 2x + 2 = \frac{4}{3} \cdot 1 + \left(-\frac{1}{2\sqrt{3}} \right) \cdot \sqrt{12} \left(x - \frac{1}{2} \right) + \frac{1}{6\sqrt{5}} \cdot 6\sqrt{5} \left(x^2 - x + \frac{1}{6} \right) . \square$$

"A man can believe a considerable deal of rubbish, and yet go about his daily work in a rational and cheerful manner." - NORMAN DOUGLAS

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