Math 308
5-5-1994

Solutions to Problem Set 9.

1. (a) Let A be an $m \times m$ matrix, B an $m \times n$ matrix, 0 the $n \times m$ zero matrix, and D an $n \times n$ matrix. Prove that the characteristic polynomial of

$$\begin{vmatrix} A & B \\ 0 & D \end{vmatrix}$$

is the product of the characteristic polynomials of A and D.

$$\begin{bmatrix} A & B \\ 0 & D \end{bmatrix} - \lambda I = \begin{bmatrix} A - \lambda I_m & B \\ 0 & D - \lambda I_n \end{bmatrix}.$$

By Problem 3 on Problem Set 8,

$$\det \begin{bmatrix} A & B \\ 0 & D \end{bmatrix} - \lambda I = \det \begin{bmatrix} A - \lambda I_m & B \\ 0 & D - \lambda I_n \end{bmatrix} = (\det(A - \lambda I_m))\,(\det(A - \lambda I_n)).$$

The last expression is the product of the characteristic polynomials of A and D. $\square$

(b) Find *by hand* the characteristic polynomial and the eigenvalues of

$$\begin{bmatrix} 2 & -2 & 0 & 0 \\ 2 & -3 & 0 & 0 \\ 0 & 0 & 4 & -3 \\ 0 & 0 & 2 & -1 \end{bmatrix}.$$

By part (a), the characteristic polynomial is the product of the characteristic polynomials for the two blocks:

$$[(\lambda - 2)(\lambda + 3) + 4]\,(\lambda - 4)(\lambda + 1) + 6] = (\lambda^2 + \lambda - 2)(\lambda^2 - 3\lambda + 2) = (\lambda + 2)(\lambda - 1)^2(\lambda - 2).$$

The eigenvalues are -2, 1 (double), and 2. $\square$

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2. A 3×3 matrix A has eigenvalues $\lambda = 1$ (double) and $\lambda = 3$. The corresponding eigenvectors are $\langle 1, 1, 1\rangle$, $\langle 1, 1, 0\rangle$, and $\langle 0, 1, 1\rangle$. Find A.

There are 3 eigenvectors, so A is diagonalizable; in fact,

$$S^{-1}AS = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix},$$

where S is the matrix whose columns are the eigenvectors.

Now

$$S = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}.$$

Hence,

$$S^{-1} = \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

Therefore,

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 1 & -1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 3 & 0 \\ -2 & 2 & 1 \end{bmatrix}. \quad \square$$

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3. The **trace** of a matrix is the sum of the entries on the main diagonal:

$$\operatorname{tr} A = \sum_{i=1}^{n} A_{ii}, \qquad A \in M_n(F).$$

(a) Let $A \in M_n(F)$. Prove that

$$|\lambda I - A| = \lambda^n - \operatorname{tr} A \lambda^{n-1} + \cdots + (-1)^n \det A.$$

The constant term of the characteristic polynomial is obtained by setting $\lambda = 0$:

$$|0 \cdot I - A| = |-A| = (-1)^n |A|.$$

The $(-1)^n$ comes from the fact that in $-A$, each row of A has been multiplied by -1, and there are n rows.

To prove that the trace gives the second term, induct on n. The result is easy for $n = 1$:

$$\det(\lambda \cdot 1 - [a]) = \lambda - a.$$

The trace of $[a]$ is a, and the coefficient of the λ^0 term is $-a$, so the formula is true.

Take $n > 1$ and assume the result is true for $n - 1$. Let A be an $n \times n$ matrix. I'll expand $|\lambda I - A|$ by cofactors of the last row. Let

$$B = \begin{bmatrix} \lambda - a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & \lambda - a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & \lambda - a_{nn} \end{bmatrix}.$$

The degree $n - 1$ terms could come from:

- $(-1)^{n+i}(-a_{ii})$ times the λ^{n-1} term in $|B(n \mid i)|$ for $i = 1, \ldots, n-1$.
- $(-1)^{n+n}(-a_{nn})$ times the λ^{n-1} term in $|B(n \mid n)|$.
- $(-1)^{n+n}\lambda$ times the λ^{n-2} term in $|B(n \mid n)|$.

There are no terms of the first kind. For in forming the $(n-1) \times (n-1)$ matrix $B(n \mid i)$, I cross out column i, which contains the $\lambda - a_{ii}$ term on the diagonal. Hence, $|B(n \mid i)|$ contains only terms of degree less than $n - 2$.

The second kind of term is $-a_{nn}\lambda^{n-1}$, since λ^{n-1} is the highest degree term in $|B(n \mid n)|$.

For the third kind of term, note that the λ^{n-2} term in $|B(n \mid n)|$ is

$$-\operatorname{tr} B(n \mid n) = -(a_{11} + a_{22} + \cdots + a_{n-1,n-1}),$$

by induction. Therefore, the λ^{n-1} term is

$$-(a_{11} + a_{22} + \cdots + a_{n-1,n-1})\lambda^{n-1}.$$

All together, the λ^{n-1} term in $|B|$ is

$$-(a_{11} + a_{22} + \cdots + a_{nn})\lambda^{n-1} = -\operatorname{tr} A \cdot \lambda^{n-1}.$$

This finishes the proof, by induction. □

(b) Use (a) to show that similar matrices have the same trace.

Similar matrices have the same characteristic polynomial, so (a) shows that they have the same trace. □

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4. Find $A, B \in M_2(\mathbf{R})$ such that A and B have the same characteristic polynomial but A and B are not similar.

The matrices

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$$

have characteristic polynomial λ^2, but they aren't similar. The first has rank 0, while the second has rank 1. □

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5. Find eigenvalues and eigenvectors for the following matrices over $\mathbb{C}$. You may use *Mathematica* to compute the eigenvalues and to solve systems of equations, but you may not use *Mathematica's* `Eigenvector` or `Eigensystem` commands (except to check your answers).

(a) $\begin{bmatrix} 1 & 2 \\ 3 & -4 \end{bmatrix}$

The characteristic polynomial is $\lambda^2 + 3\lambda - 10$. The eigenvalues are $\lambda = -5$ and $\lambda = 2$.
For $\lambda = -5$,

$$A + 5I = \begin{bmatrix} 6 & 2 \\ 3 & 1 \end{bmatrix}.$$

$\langle 1, -3 \rangle$ is an eigenvector.
For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} -1 & 2 \\ 3 & -6 \end{bmatrix}.$$

$\langle 2, 1 \rangle$ is an eigenvector. □

(b) $\begin{bmatrix} 2 & -3 \\ 0 & 2 \end{bmatrix}$

The characteristic polynomial is $\lambda^2 - 4\lambda + 4$. The eigenvalue is $\lambda = 2$ (double).
For $\lambda = 2$,

$$A - 2I = \begin{bmatrix} 0 & -3 \\ 0 & 0 \end{bmatrix}.$$

$\langle 1, 0 \rangle$ is an eigenvector. □

(c) $\begin{bmatrix} 3 & 1 & -1 \\ 0 & 3 & 2 \\ 0 & 0 & 1 \end{bmatrix}$

The characteristic polynomial is $9 - 15\lambda + 7\lambda^2 - \lambda^3$. The eigenvalues are $\lambda = 3$ (double) and $\lambda = 1$.
For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 2 \\ 0 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\langle 1, 0, 0 \rangle$ is an eigenvector.
For $\lambda = 3$,

$$A - 3I = \begin{bmatrix} 2 & 1 & -1 \\ 0 & 2 & 2 \\ 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

$\langle 1, -1, 1 \rangle$ is an eigenvector. $\square$

(d) $\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$

The characteristic polynomial is $\lambda^2 - 2\lambda + 5$. The eigenvalues are $\lambda = 1 \pm 2i$.
For $\lambda = 1 + 2i$,

$$A - (1 + 2i)I = \begin{bmatrix} -2i & -2 \\ 2 & -2i \end{bmatrix}.$$

$\langle i, 1 \rangle$ is an eigenvector.
Hence, $\langle -i, 1 \rangle$ is an eigenvector for $\lambda = 1 - 2i$. $\square$

"Ridicule often checks what is absurd, and fully as often smothers that which is noble." - Sir Walter Scott