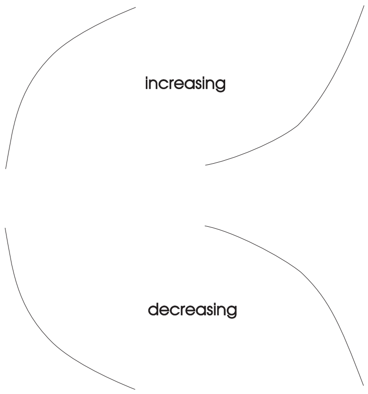
Math 123
9-14-1993

Graphing.

The first and second derivatives tell us a great deal about the graph of a function $y = f(x)$. Using the Mean Value Theorem, you can show that

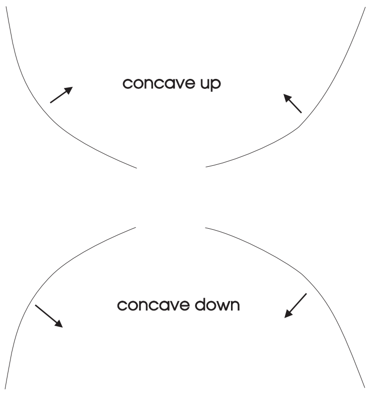
1. On an interval where $y' > 0$, the graph **increases**.

2. On an interval where $y' < 0$, the graph **decreases**.



"**Increasing**" means that the graph goes up from left to right. "**Decreasing**" means that the graph goes down from left to right.

It is not enough to know where a graph increases or decreases. The way the graph *bends* is also important. The way a graph bends is referred to as its **concavity**. A (piece of a) graph can be **concave up** or **concave down**.

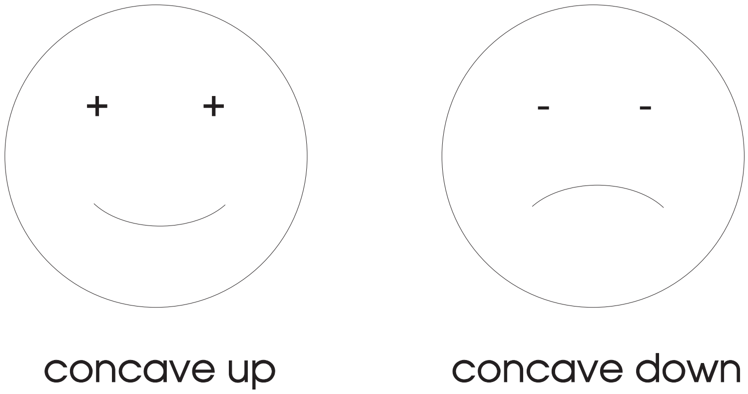


As you can see, the concavity of a graph is independent of whether the graph is increasing or decreasing.

1. On an interval where $y'' > 0$, the graph is **concave up**.

2. On an interval where $y'' < 0$, the graph **concave down**.

You can remember this by thinking of the following silly pictures:



There are other features of a graph which are important, and which we will discuss in our examples. They include x- and y-intercepts, asymptotes, local maxima and minima, and inflection points.

Examples. 1. Graph $y = 3x^4 - 4x^3 + 1$.

I will do graphing problems like this in step-by-step fashion. It makes it easier to remember the things you need to check.

The **domain** of a function is the set of "legal inputs". To determine the domain, look for values of x where the function is undefined. (For example, look for values of x that cause division by zero, or values that would produce an even root of a negative number.)

We can plug *any* value of x into $y = 3x^4 - 4x^3 + 1$ without harm. The domain is $-\infty < x < +\infty$.

The x-intercepts are the places where the graph meets the x-axis. You find them by setting $y = 0$ and solving for x. Sometimes it is difficult to solve for x; in that case, you can use a computer or calculator to approximate solutions.

In this case, $0 = y = 3x^4 - 4x^3 + 1$. By inspection — a fancy name for "trial and error" — $x = 1$ is a root. In fact, it's a double root, and $3x^4 - 4x^3 + 1 = (x-1)^2(3x^2 + 2x + 1)$ by long division. $3x^2 + 2x + 1$ has two imaginary roots, so the only x-intercept is $x = 1$.

The y-intercept is found by setting $x = 0$. For a function $y = f(x)$, there is at most one y-intercept. (Why?) In this example, $x = 0$ gives $y = 1$.

Next, I'll compute the first and second derivatives.

$$y' = 12x^3 - 12x^2 = 12x^2(x-1).$$

The final form $12x^2(x-1)$ is good because it is easy to read off the roots. *Factored forms are good for finding break points.* We will need these later to set up our sign chart.

On the other hand, to find y'', I will differentiate $12x^3 - 12x^2$ rather than $12x^2(x-1)$. *It is usually easier to differentiate a sum of terms than a product.*

Here is y'':

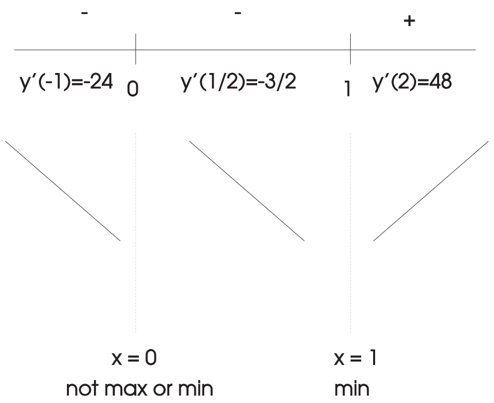
$$y'' = 36x^2 - 24x = 12x(3x-2).$$

Again, I've factored the result so I can read off the roots.

To determine where the graph increases and where it decreases, set up a **sign chart** for y'.

First, find the values of x for which $y' = 0$ or y' is undefined. (We'll refer to these values as **break points**.) Since $y' = 12x^2(x-1)$, $y' = 0$ for $x = 0$ and for $x = 1$. There are no values of x for which y' is undefined.

Put the break points onto a number line. Choose points at random in the intervals between and on either side of the break points and plug them into y'. The values you get tell you whether y' is positive or negative on the intervals.



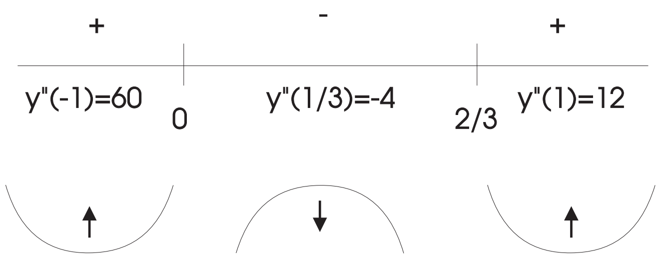
I drew downward sloping lines under the intervals where y' was negative and upward sloping lines under the intervals where y' was positive. These lines give us a schematic picture of the graph.

f decreases for $x \le 0$ and for $0 \le x \le 1$ — we can simply say $x \le 1$ — and increases for $x \ge 1$.

As a bonus, the sign chart for y' locates and classifies local maxima and minima. There is a critical point at $x = 0$, but it is neither a max nor a min since the graph decreases right through it. $x = 1$ is a min — notice how the sloping lines in the picture clearly mark it as a "valley". When $x = 1$, $y = 0$; there is a local min at $(0, 1)$. In fact, $(0, 1)$ is an absolute min.

To determine where the graph is concave up and where it is concave down, set up a **sign chart** for y''.

The break points for the sign chart are values of x where $y'' = 0$ or y'' is undefined. Since $y'' = 12x(3x - 2)$, $y'' = 0$ for $x = 0$ and for $x = \dfrac{2}{3}$. There are no values of x for which y'' is undefined.



I've drawn schematic curves ("$\smile$" and "$\frown$") under the intervals to indicate the concavity.

The graph is concave up for $x < 0$ and for $x > \dfrac{2}{3}$. It is concave down for $0 < x < \dfrac{2}{3}$.

A point where the concavity *changes* — where it goes from down to up or up to down — is called an **inflection point**. There are inflection points at $x = 0$ and at $x = \dfrac{2}{3}$.

A graph may have a **vertical asymptote** at a point where $f(x)$ is undefined. This function is defined for all x, so there are no vertical asymptotes.

To check for the presence of **horizontal asymptotes**, compute

$$\lim_{x \to +\infty} f(x) \quad \text{and} \quad \lim_{x \to -\infty} f(x).$$

For example, if $\lim_{x \to +\infty} f(x) = 17$, then the graph approaches the line $y = 17$ as it moves off to the right $(+\infty)$.

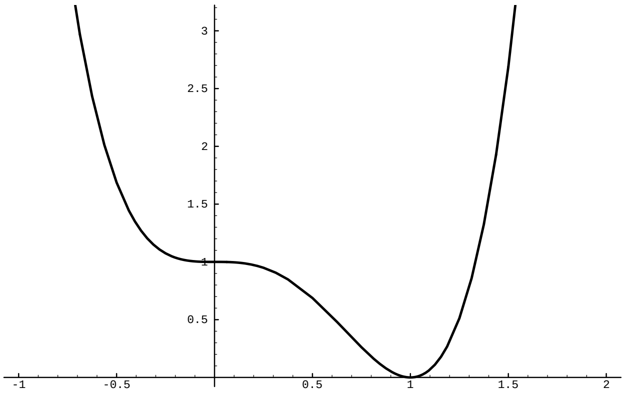
In this case,

$$\lim_{x \to +\infty} (3x^4 - 4x^3 + 1) = +\infty \quad \text{and} \quad \lim_{x \to -\infty} (3x^4 - 4x^3 + 1) = +\infty.$$

(The highest power term $3x^4$ dominates.)

The graph *rises* ($f(x) \to +\infty$) to the far left and far right. There are no horizontal asymptotes.

Finally, use the information you've obtained to sketch the graph. I've used *Mathematica* to draw the graph; it is a good check on you work, but you should also practice drawing graphs from the graphing information.



2. Graph $y = 3x^{2/3} - 2x$.

We can take the cube root of any number; the function is defined for all x.

$0 = y = 3x^{2/3} - 2x = x^{2/3}(3 - 2x^{1/3})$ gives $x = 0$ or $x = \dfrac{27}{8}$; these are the x-intercepts.

Set $x = 0$; we obtain $y = 0$. This is the y-intercept.

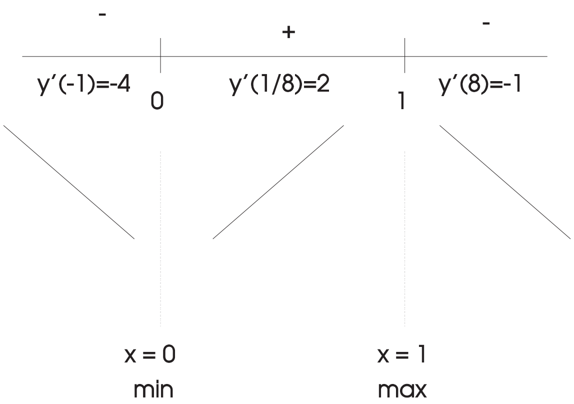
The derivatives are

$$y' = 2x^{-1/3} - 2 = x^{-1/3}(2 - 2x^{1/3}) = \frac{2 - 2x^{1/3}}{x^{1/3}},$$
$$y'' = -\frac{2}{3}x^{-4/3}.$$

In expressions with fractional exponents, you can often get a good factored form by *factoring out the largest negative power*. I did this in simplifying y': I factored out $x^{-1/3}$, then wrote it as $x^{1/3}$ on the bottom of the fraction.

Notice that in computing y'' it is easier to differentiate $2x^{-1/3} - 2$ than $\dfrac{2 - 2x^{1/3}}{x^{1/3}}$! *The factored form is good for finding break points, but the sum of terms form is better for differentiating.*

$y' = 0$ for $x = 1$. y' is undefined for $x = 0$.



The function increases for $0 \leq x \leq 1$ and decreases for $x \leq 0$ and for $x \geq 1$. There is a local min at $(0, 0)$ and a local max at $(1, 1)$.

At $x = 0$, y' is undefined but y *is* defined. This usually occurs at a place where a graph has a **corner** or a **vertical tangent**. Look at the left and right limits of y' at 0:

$$\lim_{x\to 0-} \frac{2 - 2x^{1/3}}{x^{1/3}} = -\infty \text{ and } \lim_{x\to 0+} \frac{2 - 2x^{1/3}}{x^{1/3}} = +\infty.$$

"Vertical lines have infinite slope." There is a vertical tangent at $x = 0$.

The concavity situation is very simple. $x^{-4/3}$ is an even power $((x^{1/3})^4)$, so it is always positive. Hence, $y'' = -\dfrac{2}{3}x^{-4/3}$ is always negative, and the graph is concave down everywhere.

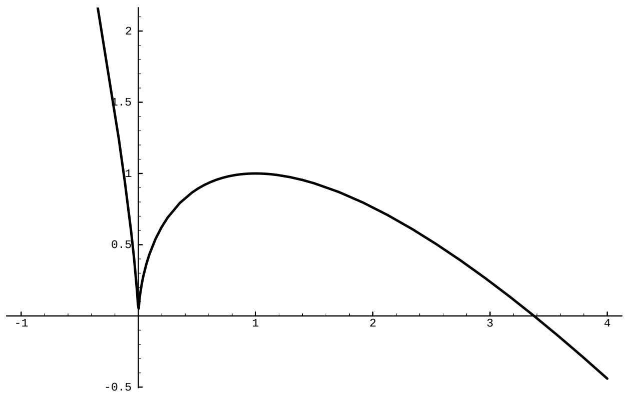
Since there can't be any concavity changes, there aren't any inflection points.

The function is defined for all x: There are no vertical asymptotes.

There are no horizontal asymptotes, because

$$\lim_{x\to+\infty}(3x^{2/3} - 2x) = +\infty \quad \text{and} \quad \lim_{x\to-\infty}(3x^{2/3} - 2x) = -\infty.$$

Here's the graph:



3. Graph $y = \dfrac{x^2+1}{x^2-1}$.

The domain is all x except $x = \pm 1$.

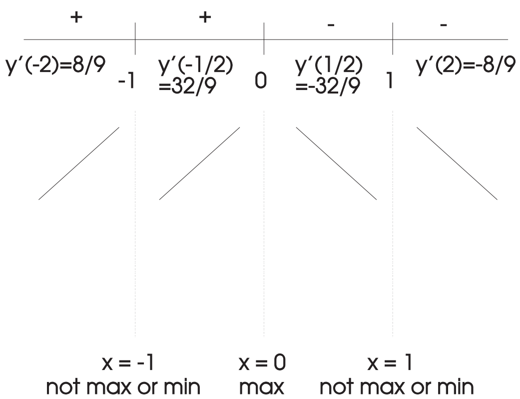
$0 = y = \dfrac{x^2+1}{x^2-1}$ gives $x^2 + 1 = 0$, which has no solutions. There are no x-intercepts.

Set $x = 0$: The y-intercept is $y = -1$.

The derivatives are

$$y' = \frac{-4x}{(x^2-1)^2}, \qquad y'' = \frac{4(3x^2+1)}{(x^2-1)^3}.$$

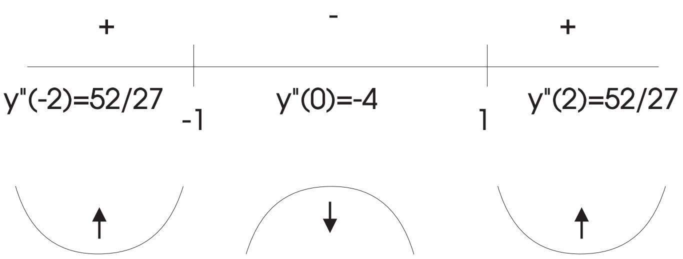
$y' = 0$ for $x = 0$; y' is undefined for $x = \pm 1$.



The graph increases for $x < -1$ and for $-1 < x < 0$. It decreases for $0 < x < 1$ and for $x > 1$.

There is a local max at $(0, -1)$.

$y'' = \dfrac{4(3x^2+1)}{(x^2-1)^3}$ is never 0; it is undefined for $x = \pm 1$.



The graph is concave up for $x < -1$ and for $x > 1$. The graph is concave down for $-1 < x < 1$. The function is undefined at $x = \pm 1$, so these do not count as inflection points.

The function is undefined at $x = \pm 1$, so there may be vertical asymptotes at these points. Compute the left- and right-hand limits at -1 and 1:

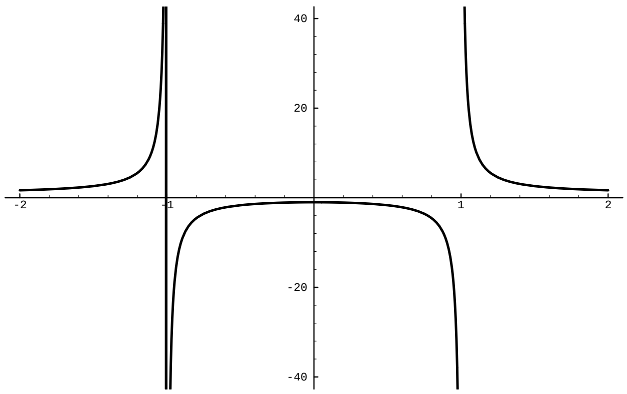
$$\lim_{x \to -1+} \frac{x^2+1}{x^2-1} = +\infty, \quad \lim_{x \to -1-} \frac{x^2+1}{x^2-1} = -\infty,$$
$$\lim_{x \to 1+} \frac{x^2+1}{x^2-1} = +\infty, \quad \lim_{x \to 1-} \frac{x^2+1}{x^2-1} = -\infty.$$

The lines $x = 1$, $x = -1$ are vertical asymptotes.

Since

$$\lim_{x \to +\infty} \frac{x^2+1}{x^2-1} = 1, \quad \lim_{x \to -\infty} \frac{x^2+1}{x^2-1} = 1,$$

the graph is asymptotic to $y = 1$ as $x \to +\infty$ and as $x \to -\infty$.



Problems on Graphing.

1. Graph $y = 2x^{3/2} - 6x^{1/2}$.

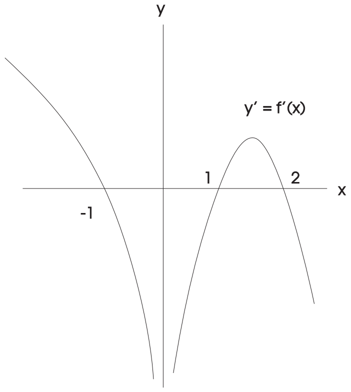
2. Graph $y = \dfrac{x}{\sqrt{x^2+7}}$.

3. Graph $y = 5x^{2/5} + \dfrac{5}{7}x^{7/5}$.

4. Graph $y = \dfrac{2x}{x^2-1}$.

5. Sketch the graph of $y = |x^2 - 6x + 5|$ by first sketching the graph of $y = x^2 - 6x + 5$.

6. The function $y = f(x)$ is defined for all x. The graph of its derivative $y' = f'(x)$ is shown below:



Sketch the graph of $y = f(x)$.

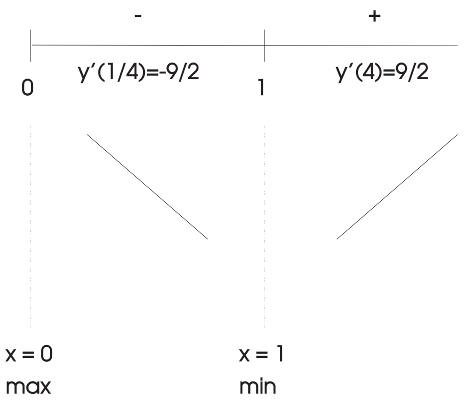
Solutions to the Problems on Graphing.

1. Graph $y = 2x^{3/2} - 6x^{1/2}$.

The domain is $x \geq 0$.
$0 = y = 2x^{3/2} - 6x^{1/2} = 2x^{1/2}(x-3)$ gives the x-intercepts $x = 0$ and $x = 3$.
Set $x = 0$; the y-intercept is $y = 0$.
The derivatives are

$$y' = 3x^{1/2} - 3x^{-1/2} = \frac{3(x-1)}{x^{1/2}},$$
$$y'' = \frac{3}{2}x^{-1/2} + \frac{3}{2}x^{-3/2} = \frac{3}{2} \cdot \frac{x+1}{x^{3/2}}.$$

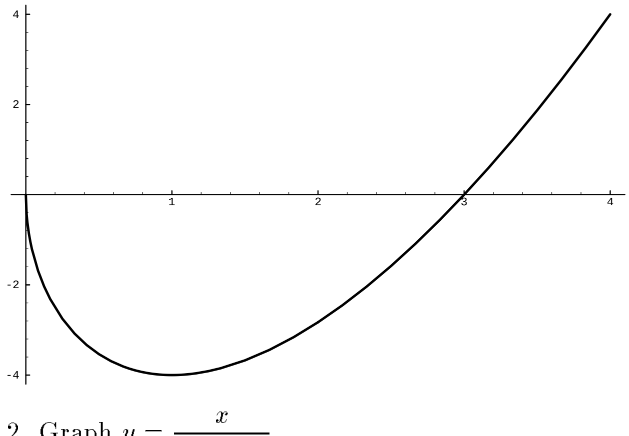
$y' = \dfrac{3(x-1)}{x^{1/2}} = 0$ for $x = 1$; y' is undefined for $x \leq 0$. ($y = 0$ is an endpoint of the domain.)



The graph decreases for $0 \leq x \leq 1$ and increases for $x \geq 1$.
There is a local (endpoint) max at $(0, 0)$ and a local (absolute) min at $(1, -4)$.
$y'' = \dfrac{3}{2} \cdot \dfrac{x+1}{x^{3/2}}$ is always positive, since the factors $\dfrac{3}{2}$, $x + 1$, and $x^{3/2}$ are always positive. (Remember that the domain is $x \geq 0$!) Hence, the graph is always concave up, and there are no inflection points.
There are no vertical asymptotes. The domain ends at $x = 0$, but $f(0) = 0$.
Since

$$\lim_{x \to +\infty} (2x^{3/2} - 6x^{1/2}) = +\infty,$$

the graph rises on the far right. (It does not make sense to compute $\lim_{x \to -\infty}(2x^{3/2} - 6x^{1/2})$. Why?) Hence, there are no horizontal asymptotes.



2. Graph $y = \dfrac{x}{\sqrt{x^2 + 7}}$.

The function is defined for all x.

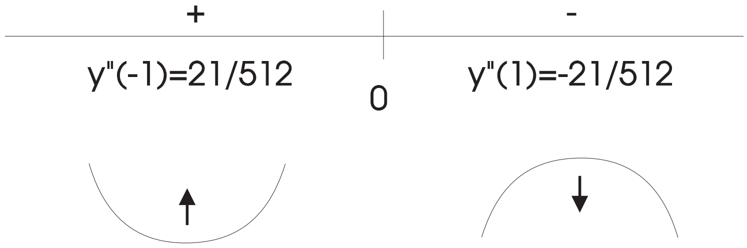
The x-intercept is $x = 0$; the y-intercept is $y = 0$.

The derivatives are

$$y' = \frac{\sqrt{x^2+7} - \frac{x^2}{\sqrt{x^2+7}}}{x^2+7} = \frac{7}{(x^2+7)^{3/2}},$$
$$y'' = \frac{-21x}{(x^2+7)^3}.$$

Since $(x^2+7)^{3/2} > 0$ for all x, y' is always positive. Hence, the graph is always increasing. There are no maxima or minima.

$y'' = \dfrac{-21x}{(x^2+7)^3} = 0$ for $x = 0$; y'' is defined for all x.



The graph is concave up for $x < 0$ and concave down for $x > 0$. $x = 0$ is a point of inflection.

There are no vertical asymptotes, since the function is defined for all x.

Clearly,

$$\lim_{x\to+\infty} \frac{x}{\sqrt{x^2+7}} = 1,$$

so the graph is asymptotic to $y = 1$ as $x \to +\infty$.

You may find it surprising that

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+7}} = -1.$$

Actually, it is no surprise if you think about it. Since $x \to -\infty$, x is taking on *negative* values. The numerator x is negative, but the denominator $\sqrt{x^2+7}$ is always positive, by definition. Hence, the limit must be negative, or at least not positive.

Algebraically, this is a result of the fact that

$$\sqrt{u^2} = |u|,$$

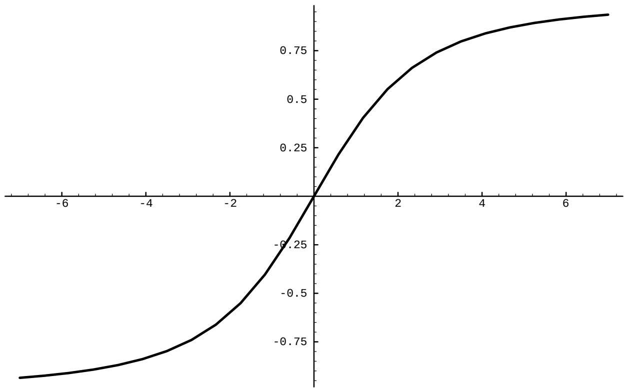
and in turn $|u| = -u$ if u is negative. (The absolute value of a negative number is the negative of the number.)

Here is the computation in more detail:

$$\lim_{x\to-\infty} \frac{x}{\sqrt{x^2+7}} = \lim_{x\to-\infty} \frac{\frac{1}{x}\cdot x}{\frac{1}{x}\cdot\sqrt{x^2+7}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{\frac{1}{x^2}\cdot(x^2+7)}} = \lim_{x\to-\infty} \frac{1}{-\sqrt{1+\frac{7}{x^2}}} = -1.$$

Notice the negative sign that appeared when I pushed the $\dfrac{1}{x}$ into the square root. This is a result of the algebraic fact I mentioned above.

Anyway, the graph is asymptotic to $y = -1$ as $x \to -\infty$.



3. Graph $y = 5x^{2/5} + \frac{5}{7}x^{7/5}$.

The function is defined for all x.

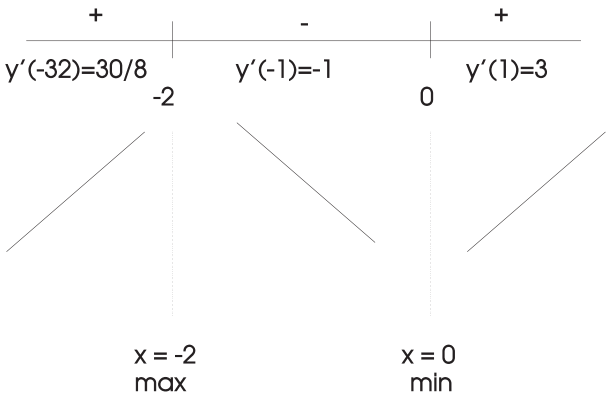
$0 = y = 5x^{2/5} + \frac{5}{7}x^{7/5} = 5x^{2/5}(1 + \frac{1}{7}x)$ for $x = 0$ and $x = -7$. These are the x-intercepts.

Set $x = 0$; the y-intercept is $y = 0$.

The derivatives are

$$y' = 2x^{-3/5} + x^{2/5} = x^{-3/5}(2 + x) = \frac{x+2}{x^{3/5}},$$
$$y'' = -\frac{6}{5}x^{-8/5} + \frac{2}{5}x^{-3/5} = \frac{1}{5}x^{-8/5}(-6 + 2x) = \frac{1}{5} \cdot \frac{2(x-3)}{x^{8/5}}.$$

$y' = \frac{x+2}{x^{3/5}} = 0$ for $x = -2$. y' is undefined at $x = 0$.



The graph increases for $x \le -2$ and for $x \ge 0$. The graph decreases for $-2 \le x \le 0$.

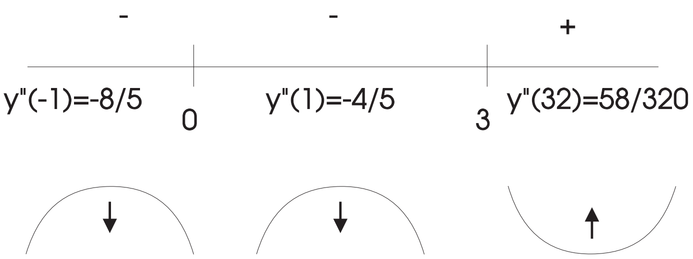
There is a local max at $x = -2$; the approximate y-value is 4.71253. There is a local min at $(0, 0)$.

Note that

$$\lim_{x \to 0-} \frac{x+2}{x^{3/5}} = -\infty \text{ and } \lim_{x \to 0+} \frac{x+2}{x^{3/5}} = +\infty.$$

There is a vertical tangent at the origin.

$y'' = \frac{1}{5} \cdot \frac{2(x-3)}{x^{8/5}} = 0$ at $x = 3$; y'' is undefined at $x = 0$.

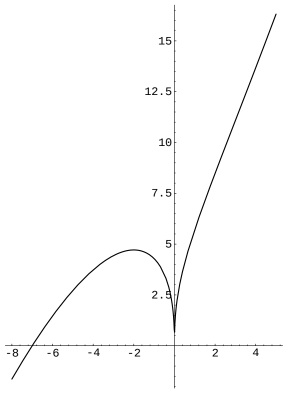


The graph is concave down for $x < 0$ and for $0 < x < 3$. The graph is concave up for $x > 3$. There is a point of inflection at $x = 3$.

There are no vertical asymptotes.

$$\lim_{x\to+\infty}(5x^{2/5} + \frac{5}{7}x^{7/5}) = +\infty \text{ and } \lim_{x\to-\infty}(5x^{2/5} + \frac{5}{7}x^{7/5}) = -\infty.$$

The graph falls to $-\infty$ on the far left and rises to $+\infty$ on the far right.



4. Graph $y = \dfrac{2x}{x^2 - 1}$.

The domain is all x except $x = \pm 1$.

The x-intercept is $x = 0$; the y-intercept is $y = 0$.

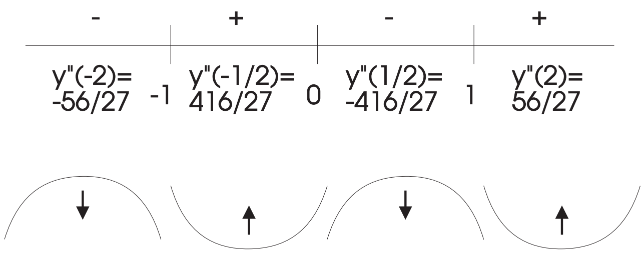
The derivatives are

$$y' = \frac{(x^2-1)(2) - (2x)(2x)}{(x^2-1)^2} = \frac{-2(x^2+1)}{(x^2-1)^2},$$

$$y'' = -2 \cdot \frac{(x^2-1)^2(2x) - (x^2+1)(2)(x^2-1)(2x)}{(x^2-1)^4} = \frac{4x(x^2+3)}{(x^2-1)^3}.$$

$y' = \dfrac{-2(x^2+1)}{(x^2-1)^2}$ is always negative: -2 is negative, while x^2+1 and $(x^2-1)^2$ are always positive. The graph is decreasing everywhere; there are no local maxima or minima.

$y'' = \dfrac{4x(x^2+3)}{(x^2-1)^3} = 0$ for $x = 0$. y'' is undefined at $x = \pm 1$.



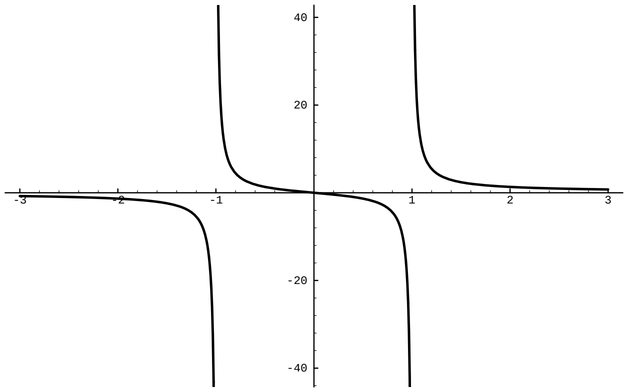
The graph is concave up for $-1 < x < 0$ and for $x > 1$. The graph is concave down for $x < -1$ and for $0 < x < 1$. $x = 0$ is the only inflection point.

There are vertical asymptotes at $x = \pm 1$. In fact,

$$\lim_{x\to 1+} \frac{2x}{x^2-1} = +\infty, \quad \lim_{x\to 1-} \frac{2x}{x^2-1} = -\infty,$$
$$\lim_{x\to -1+} \frac{2x}{x^2-1} = +\infty, \quad \lim_{x\to -1-} \frac{2x}{x^2-1} = -\infty.$$

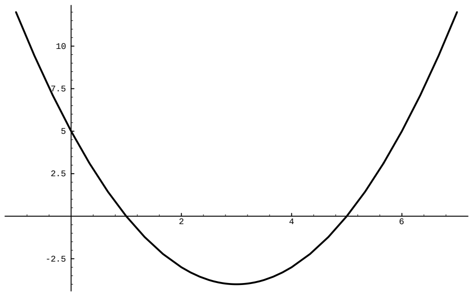
The graph is asymptotic to $y = 0$ as $x \to +\infty$ and as $x \to -\infty$:

$$\lim_{x\to +\infty} \frac{2x}{x^2-1} = 0 \text{ and } \lim_{x\to -\infty} \frac{2x}{x^2-1} = 0.$$

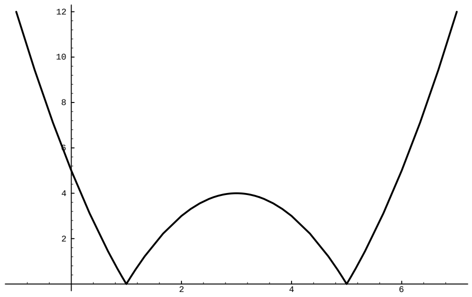


5. Sketch the graph of $y = |x^2 - 6x + 5|$ by first sketching the graph of $y = x^2 - 6x + 5$.

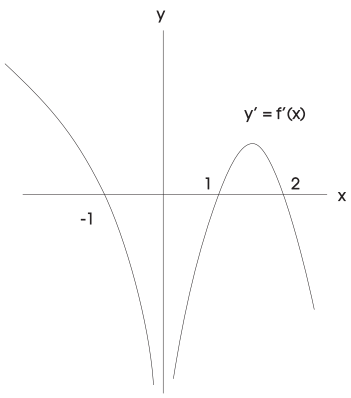
$y = x^2 - 6x + 5 = (x-1)(x-5)$ is a parabola with roots at $x = 1$ and at $x = 5$, opening upward:



The absolute value function leaves the positive parts alone and reflects the negative parts about the x-axis. Hence, the graph of $y = |x^2 - 6x + 5|$ is



6. The function $y = f(x)$ is defined for all x. The graph of its derivative $y' = f'(x)$ is shown below:



Sketch the graph of $y = f(x)$.

Since

$$\lim_{x \to 0+} f'(x) = \lim_{x \to 0-} f(x) = -\infty,$$

and since f is defined for all x (including $x = 0$), there is a vertical tangent at the origin.

Wherever $y' = 0$ — i.e. wherever y' crosses the x-axis, we have a critical point. In fact, there are local maxima at $x = -1$ and at $x = 2$, and there is a local minimum at $x = 1$.