# Variation of parameters.

Consider the second-order linear equation

$$y'' + p(x)y' + q(x)y = f(x).$$

Suppose that you know two *independent* solutions $y_1$, $y_2$, to the corresponding homogeneous equation

$$y'' + p(x)y' + q(x)y = 0.$$

(This will be the case, for instance, if the homogeneous equation has constant coefficients.) I'll try to find a solution to the original equation of the form

$$y = u_1 y_1 + u_2 y_2.$$

The problem is to determine the functions $u_1$ and $u_2$. This method is called **variation of parameters**. Notice that this looks like **reduction of order**, except that I have *two* homogeneous solutions instead of one.

If $y = u_1 y_1 + u_2 y_2$ is to be a solution, it must satisfy the original equation. Compute the derivatives:

$$y' = u_1' y_1 + u_1 y_1' + u_2' y_2 + u_2 y_2',$$

$$y'' = u_1'' y_1 + 2u_1' y_1' + u_1 y_1'' + u_2'' y_2 + 2u_2' y_2' + u_2 y_2''.$$

Plug everything into the original equation:

$$f(x) = y'' + py' + qy =$$

$$\begin{array}{cccccccccccc}
u_1''y_1 & + & 2u_1'y_1' & + & u_1y_1'' & + & u_2''y_2 & + & 2u_2'y_2' & + & u_2y_2'' & \\
 & + & pu_1'y_1 & + & pu_1y_1' & & & + & pu_2'y_2 & + & pu_2y_2' & \\
 & & & + & qu_1y_1 & & & & & + & qu_2y_2 & = \\
u_1''y_1 & + & (2u_1'y_1' + pu_1'y_1) & & & + & u_2''y_2 & + & (2u_2'y_2' + pu_2'y_2) & & &
\end{array}$$

($y_1'' + py_1' + qy_1 = 0$ and $y_2'' + py_2' + qy_2 = 0$ since $y_1$ and $y_2$ are homogeneous solutions.)
In other words,

$$u_1''y_1 + u_1'(2y_1' + py_1) + u_2''y_2 + u_2'(2y_2' + py_2) = f(x).$$

Regroup the terms:

$$(u_1''y_1 + u_2''y_2) + 2(u_1'y_1' + u_2'y_2') + p(u_1'y_1 + u_2'y_2) = f(x).$$

Next, write $2(u_1'y_1' + u_2'y_2')$ as $(u_1'y_1' + u_2'y_2') + (u_1'y_1' + u_2'y_2')$:

$$(u_1''y_1 + u_2''y_2 + u_1'y_1' + u_2'y_2') + (u_1'y_1' + u_2'y_2') + p(u_1'y_1 + u_2'y_2) = f(x).$$

But by the Product Rule,

$$u_1''y_1 + u_2''y_2 + u_1'y_1' + u_2'y_2' = (u_1'y_1 + u_2'y_2)'.$$

So the equation is

$$(u_1'y_1 + u_2'y_2)' + (u_1'y_1' + u_2'y_2') + p(u_1'y_1 + u_2'y_2) = f(x).$$

Observe that the expression $u_1'y_1 + u_2'y_2$ appears twice in this equation.
After all that algebra, it might be good to step back and think about what's going on!

I'm trying to solve for $u_1$ and $u_2$. The only condition (equation) I have to determine them is the one above. But to solve for *two* things you normally need *two* conditions. Therefore, I need to impose another condition on $u_1$ and $u_2$.

I have the freedom to impose any reasonable condition I wish. Given the repeated appearance of $u_1'y_1 + u_2'y_2$ in the last equation, I'll use

$$u_1'y_1 + u_2'y_2 = 0.$$

Plug this into the equation above; it becomes

$$u_1'y_1' + u_2'y_2' = f.$$

Thus, I need to solve the following equations simultaneously for $u_1$ and $u_2$:

$$u_1'y_1 + u_2'y_2 = 0, \quad u_1'y_1' + u_2'y_2' = f(x).$$

It's easy to remember the equations if they're written in matrix form:

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f \end{bmatrix}.$$

*I'll take this as the starting point in our variation of parameters problems.* The homogeneous solutions $y_1$ and $y_2$ are inserted on the left side, and the "forcing function" $f(x)$ is inserted on the right. The problem reduces to solving for $u_1$ and $u_2$. You can do this by inverting the matrix

$$\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}.$$

The formula for the inverse of a 2 matrix is

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

Note that for the matrix $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}$, the term $ad - bc$ is just the Wronskian

$$W(y_1, y_2) = y_1y_2' - y_2y_1'.$$

Since $y_1$ and $y_2$ are assumed to be independent, the Wronskian is nonzero, and the division makes sense.

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**Example.** Solve

$$(D^2 + 1)y = \sec x \tan x.$$

I can't use undetermined coefficients here, because $\sec x \tan x$ is not a solution to a constant coefficient homogeneous linear equation. I'll use variation of parameters instead.

$y_1 = \cos x$ and $y_2 = \sin x$ are independent homogeneous solutions; I'll look for a solution of the form $y = u_1 \cos x + u_2 \sin x$. Write down the variation of parameters equation and solve:

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \sec x \tan x \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ \sec x \tan x \end{bmatrix} = \begin{bmatrix} -(\tan x)^2 \\ \tan x \end{bmatrix}.$$

Now solve for $u_1$ and $u_2$:

$$u_1' = -(\tan x)^2 = 1 - (\sec x)^2, \quad u_1 = x - \tan x + c_1,$$

$$u_2' = \tan x, \quad u_2 = \ln|\sec x| + c_2.$$

The solution is

$$y = x\cos x - \sin x + (\sin x)\ln|\sec x| + c_1\cos x + c_2\sin x = x\cos x + (\sin x)\ln|\sec x| + c_1\cos x + c_3\sin x.$$

(In the last step, the "$-\sin x$" term was "absorbed" into the "$c_2 \sin x$" term, the arbitrary constant changing appropriately.) □

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**Warning:** The variation of parameters matrix equation was derived from a second order equation in the form

$$y'' + p(x)y' + q(x)y = r(x).$$

That is, the coefficient of $y''$ is 1.

Therefore, if your equation has some $x$-junk multiplying $y''$, you must divide through by the $x$-junk before you start. □

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## Problems on Variation of Parameters.

1. Solve $(D^2 - 1)y = e^{-2x}\sin e^{-x}$.

2. (a) Verify that $x$ and $e^x$ are homogeneous solutions for

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}.$$

(b) Use variation of parameters to solve

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}.$$

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## Solutions to the Problems on Variation of Parameters.

1. Solve $(D^2 - 1)y = e^{-2x}\sin e^{-x}$.

$y_1 = e^x$ and $y_2 = e^{-x}$ are independent homogeneous solutions; I'll look for a solution of the form $y = u_1e^x + u_2e^{-x}$. Write down the variation of parameters equation and solve:

$$\begin{bmatrix} e^x & e^{-x} \\ e^x & -e^{-x} \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ e^{-2x}\sin e^{-x} \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -e^{-x} & -e^{-x} \\ -e^x & e^x \end{bmatrix} \begin{bmatrix} 0 \\ e^{-2x}\sin e^{-x} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} -e^{-3x}\sin e^{-x} \\ e^{-x}\sin e^{-x} \end{bmatrix}.$$

Now solve for $u_1$ and $u_2$:

$$u_1' = \frac{1}{2}e^{-3x}\sin e^{-x}, \quad u_1 = -\cos e^{-x} + \frac{1}{2}e^{-2x}\cos e^{-x} - e^{-x}\sin e^{-x} + c_1,$$

$$u_2' = -\frac{1}{2}e^{-x}\sin e^{-x}, \quad u_2 = -\frac{1}{2}\cos e^{-x} + c_2.$$

The solution is

$$y = -e^x \cos e^{-x} + \frac{1}{2}e^{-x}\cos e^{-x} - \sin e^{-x} - \frac{1}{2}e^{-x}\cos e^{-x} + c_1 e^x + c_2 e^{-x} = c_1 e^x + c_2 e^{-x} - e^x \cos e^{-x} - \sin e^{-x}. \square$$

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2. (a) Verify that $x$ and $e^x$ are homogeneous solutions for

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}.$$

$$y_1 = x, \quad y_1' = 1, \quad y_1'' = 0, \quad (1-x)y_1'' + xy_1' - y_1 = 0 + x - x = 0.$$

$x$ is a homogeneous solution.

$$y_2 = e^x, \quad y_2' = e^x, \quad y_2'' = e^x, \quad (1-x)y_2'' + xy_2' - y_2 = (1-x)e^x + xe^x - e^x = 0.$$

$e^x$ is a homogeneous solutions. (You can check using the Wronskian that they're also independent.) $\square$

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(b) Use variation of parameters to solve

$$(1-x)y'' + xy' - y = 2(x-1)^2 e^{-x}.$$

$y_1 = x$ and $y_2 = e^x$ are independent homogeneous solutions; I'll look for a solution of the form $y = u_1 x + u_2 e^x$.

First, I have to put the equation into the proper form: divide through by $1-x$ to obtain

$$y'' + \frac{x}{1-x}y' - \frac{y}{1-x} = -2(x-1)e^{-x}.$$

Now write down the variation of parameters equation and solve:

$$\begin{bmatrix} x & e^x \\ 1 & e^x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ -2(x-1)e^{-x} \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{x-1}e^{-x} \begin{bmatrix} e^x & -e^x \\ -1 & x \end{bmatrix} \begin{bmatrix} 0 \\ -2(x-1)e^{-x} \end{bmatrix} = \begin{bmatrix} 2e^{-x} \\ -2xe^{-2x} \end{bmatrix}.$$

Solve for $u_1$ and $u_2$:

$$u_1' = 2e^{-x}, \quad u_1 = -2e^{-x} + c_1,$$

$$u_2' = -2xe^{-2x}, \quad u_2 = xe^{-2x} + \frac{1}{2}e^{-2x} + c_2.$$

The solution is

$$y = -2xe^{-x} + xe^{-x} + \frac{1}{2}e^{-x} + c_1 x + c_2 e^x = -xe^{-x} + \frac{1}{2}e^{-x} + c_1 x + c_2 e^x. \square$$

 