PROBLEMS IN ALGEBRAIC COMBINATORICS

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Abstract: This is a list of open problems, mainly in graph theory and all with an algebraic flavour. Except for 6.1, 7.1 and 12.2 they are either folklore, or are stolen from other people.

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1. Moore Graphs

We define a *Moore Graph* to be a graph with diameter d and girth $2d+1$. Somewhat surprisingly, any such graph must necessarily be regular (see [42]) and, given this, it is not hard to show that any Moore graph is distance regular. The complete graphs and odd cycles are trivial examples of Moore graphs. The Petersen and Hoffman-Singleton graphs are non-trivial examples. These examples were found by Hoffman and Singleton [23], where they showed that if X is a k-regular Moore graph with diameter two then $k \in \{2, 3, 7, 57\}$. This immediately raises the following question:

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1.1 Problem. *Is there a regular graph with valency 57, diameter two and girth five?*

We summarise what is known. Bannai and Ito [4] and, independently, Damerell [12] showed that a Moore graph has diameter at most two. (For an exposition of this, see Chapter 23 in Biggs [7].) Aschbacher [2] proved that a Moore graph with valency 57 could not be distance transitive and G. Higman (see [9]) proved that it could not even be vertex transitive.

By either a square-counting or an interlacing argument, one can show that the maximum number of vertices in an independent set in the Hoffman-Singleton graph is 15. If S is an independent set of size 15 in this graph then each vertex not in S is adjacent to exactly three vertices in S, and so the graph induced by the vertices not in S is 4-regular. This leads to a construction of the Hoffman-Singleton graph. Let G be the graph formed by the 35 triples from a set of seven points, with two triples adjacent if they are disjoint. (This is the odd graph $O(4)$, with diameter three.) Call a set of seven triples such that any pair meet in exactly one point a *heptad*. It is not too hard to show that there are exactly 30 heptads, all equivalent under the action of $\mathrm{Sym}(7)$ and falling into two orbits of length 15 under $\mathrm{Alt}(7)$. Choose one of these two orbits and then extend G to a graph H by adding 15 new vertices, each adjacent to all seven vertices in a heptad in the selected orbit. Then H is the Hoffman-Singleton graph.

Note that we may view the vertices in S as points and the vertices not in S as lines, with a point and line incident if the corresponding vertices are adjacent. This gives us a 2-$(15, 7, 1)$ design and in the construction above this design is the design of points and lines in $PG(3, 2)$. Now consider a possible Moore graph with valency 57. In this case an independent set has at most 400 vertices. If S is an independent set of cardinality then the incidence structure formed by the vertices in S and the vertices not in S is a 2-$(400, 8, 1)$ design. The points and lines of $PG(3, 7)$ form a design with these parameters.

The Hoffman-Singleton graph contains many copies of the Petersen graph. It is easy to show that there are subgraphs in it isomorphic to Petersen's graph with one edge deleted, and it can be shown that any such subgraph must actually induce a copy of the Petersen graph. (See [10: Theorem 6.6], where this is used to show that the Hoffman-Singleton graph is unique, i.e., it is the only graph of diameter two, girth five and valency seven.) As far as I know, it has not been proved that Moore graph with valency 57 must contain even one copy of the Petersen graph (to say nothing of the Hoffman-Singleton graph).

2. Triangle-free Strongly Regular Graphs

A graph is *strongly regular* if it is not complete or empty and the number of common neighbours of two vertices is determined by whether they are equal, adjacent or neither equal nor adjacent. An $(n, k; a, c)$ strongly regular graph is a k-regular graph on n vertices such that any pair of adjacent vertices has exactly a common neighbours while a pair of distinct non-adjacent vertices has exactly c common neighbours. We are concerned with strongly regular graphs with no triangles, i.e., with $a = 0$. Any Moore graph with diameter two is an example.

Three have already appeared—the cycle on five vertices, the Petersen graph and the Hoffman-Singleton graph—but only four more are known. We describe them. The first is the Clebsch graph, which we build from Petersen's graph.

We may view the vertices of the Petersen graph as the unordered pairs from the set

$$F := \{0, 1, 2, 3, 4\},$$

where two unordered pairs are adjacent if and only if they are disjoint. It is not hard to show that the maximum size of an independent set in Petersen's graph is four, and that any such set consists of the four pairs containing a given point from our F. Let S_i be the independent set formed of the four pairs containing i. Now construct a graph C as follows. If P denotes the vertex set of the Petersen graph, vertex set of C is

$$\infty,\ F,\ P.$$

The vertex ∞ is adjacent to each of the points of F and the vertex i in F is adjacent to all vertices in S_i. Thus C is a 5-regular triangle-free graph on 16 vertices, and it is not difficult to show that it is strongly regular.

The Higman-Sims graph is also very easy to construct. Let W_{22} be the Witt design on 23 points. This is a 3-$(22, 6, 1)$ design with 77 blocks. Let V be the point set of W_{22} and let $\mathcal{B}$ be its block set. The vertex set of the Higman-Sims graph is the set

$$\infty \cup V \cup \mathcal{B}.$$

The adjacencies are as follows. The vertex ∞ is adjacent to all vertices in V and each block is adjacent to the six points in V which lie in it, and to all the blocks in $\mathcal{B}$ which are disjoint from it. With some effort it can be shown that this is a $(100, 22; 0, 6)$ strongly regular graph.

It is possible to partition the vertices of the Higman-Sims graph into two sets of size 50, with the subgraph induced by each set isomorphic the Hoffman-Singleton graph. (See, e.g., Exercise 2 in Chapter 8 of [10] or Chapter VI of [41].)

The graph induced by the vertices at distance two from a chosen vertex in the Higman-Sims graph form a subgraph isomorphic to the complement of the block graph of W_{22}. (The block graph of a design has the blocks of the design for vertices, with two blocks adjacent if and only if the have a point in common.) Thus the complement of the block graph of W_{22} is triangle-free. It is also a $(77, 16; 0, 4)$ strongly regular graph. (This follows from standard results on quasi-symmetric designs.) The 21 blocks in W_{22} containing a given point form an incidence structure isomorphic to the projective plane of order four. The remaining 56 blocks form another quasi-symmetric design and the complement of its block graph is a $(56, 10; 0, 2)$ strongly regular graph, known as the *Gewirtz graph*.

Now we have seen seven triangle-free strongly regular graphs, which leads naturally to the question for this section.

2.1 Problem. *Is there an eighth triangle-free strongly regular graph?*

Biggs [5: Section 4.6]] shows that if a $(n, k; 0, c)$ strongly regular graph exists and $c \notin \{2, 4, 6\}$ then k is bounded by a function of c. This bounds n too, since

$$n = 1 + k + \frac{k(k-1)}{c}.$$

The smallest open case appears to be the existence of a strongly regular graph with parameters $(162, 21; 0, 3)$.

Triangle-free strongly regular graphs are of interest in knot theory. For more information about the connection see, e.g., Jaeger [24]. Unfortunately for the knot theorists the strongly regular graphs they need must not only be triangle-free, they should also be "formally self-dual". For what this means see [24], (or [17: p. 249]); this extra condition does imply that the set of vertices at distance two from a fixed vertex must also be a strongly regular graph. The Higman-Sims graph is formally self-dual.

3. Equiangular Lines

A set of lines through the origin in $\mathbf{R}^n$ is *equiangular* if the angle between any two lines is the same. Our general problem is to determine the maximum size of a set of equiangular lines in $\mathbf{R}^m$. The diagonals of the icosahedron provide a set of six equiangular lines in $\mathbf{R}^3$.

Let $\mathcal{L}$ be a set of equiangular lines in $\mathbf{R}^m$ and let $x_1, \dots, x_m$ be a set of unit vectors such that x_i spans the i-th line of $\mathcal{L}$. Let U be the matrix with i-column equal to x_i and let γ denote $|x_i^T x_j|^{-1}$, for $i \neq j$. Then

$$U^T U = I + \gamma^{-1} S$$

where S is a symmetric matrix with all diagonal entries equal to zero, all off-diagonal entries equal to 1 or -1, rank m and least eigenvalue $-\gamma$. Since S is an integer matrix this implies that γ is an algebraic integer. Further, if γ is not rational then its multiplicity $n - m$ can be at most $n/2$. Thus γ must be rational if $n > 2m$. Since the only rational algebraic integers are the plain old-fashioned integers, we deduce that if $n > 2m$ then γ is an integer. In fact γ must be an odd integer, as we now show.

To see this let A be $\frac{1}{2}(S+J-I)$. Then A is a symmetric 01-matrix and $S = 2A+I-J$. If $n - m > 2$ then γ is an eigenvalue of $S + J$ (with multiplicity at least $n - m - 1$. Hence $(\gamma + 1)/2$ is a rational eigenvalue of A. Since A is an integer matrix and γ is rational, this implies that $(\gamma + 1)/2$ is an integer, and γ must be an odd integer.

Let X_i be the matrix xx^T, which represents orthogonal projection on the line spanned by x_i. (Note that replacing x_i by $-x_i$ does not change X_i.) Finally suppose that the square of the cosine of the angle between any two distinct lines in $\mathcal{L}$ is λ. The matrices X_i lie in the vector space of all symmetric $m \times m$ matrices, which has dimension $\binom{m+1}{2}$. The mapping $(A, B) \to \operatorname{tr} AB$ is an inner product on this space and the Gram matrix of $X_1, \dots, X_n$ with respect to this inner product is

$$(1 - \lambda)I + \lambda J.$$

Since $\lambda < 1$ this is the sum of a positive definite and a positive semidefinite matrix. Hence it is positive definite and therefore invertible. Consequently the matrices $X_1, \dots, X_n$ are linearly independent and therefore $n \leq \binom{m+1}{2}$. Before discussing how good this bound is, we examine what happens in the case of equality.

If $n = \binom{m+1}{2}$ then $X_1, \dots, X_n$ is a basis for the space of symmetric $m \times m$ matrices. Hence there are constants c_i such that

$$I = \sum_{i=1}^{n} c_i X_i. \tag{3.1}$$

Now

$$\mathrm{tr}(X_i - \lambda I)X_j = \begin{cases} 1-\lambda, & \text{if } i = j; \\ 0, & \text{otherwise} \end{cases}$$

and therefore Equation (3.1) yields that

$$\mathrm{tr}(X_i - \lambda I) = c_i(1-\lambda).$$

Thus $c_i = (1-m\lambda)/(1-\lambda)$ and taking the trace of both sides of Equation (3.1) yields that

$$n = \frac{m(1-\lambda)}{1-m\lambda}. \tag{3.2}$$

Substituting $n = \binom{m+1}{2}$ here and solving for λ yields $\lambda = (m+2)^{-1}$, with the consequence that $m+2$ must be the square of an odd integer when $m \geq 6$.

Examples of sets of $\binom{m+1}{2}$ equiangular lines in $\mathbf{R}^m$ are known to exist, and be unique, when $m = 2, 3, 7$ and $m = 23$. (When $m = 2$ we may take the diagonals of a regular hexagon and, when $m = 3$, the diagonals of a regular isohedron. For the remaining two cases, see pages 129–130 and pages 166–167 in [40].)

3.1 Problem. *Is there a set of $\binom{m+1}{2}$ equiangular lines in $\mathbf{R}^m$ when $m > 23$?*

Infinitely many examples of sets of equiangular lines with cardinality of order $m^{3/2}$ are known [40: Theorem 10.5]; it may be that the $\binom{m+1}{2}$ bound is not even asymptotically correct.

4. Two-graphs

There is another bound on sets of equiangular lines. Consider the matrix S above. Its least eigenvalue is $-\gamma$, and this eigenvalue has multiplicity $n-m$. Let $\theta_1, \ldots, \theta_m$ be its remaining eigenvalues. Since $\mathrm{tr}\, S = 0$,

$$(n-m)\gamma = \sum_i \theta_i$$

and, since $\mathrm{tr}\, S^2 = n(n-1)$,

$$n(n-1) - (n-m)\gamma^2 = \sum_i \theta_i.$$

These two equations imply that

$$\frac{n(n-1)-(n-m)\gamma^2}{m} \geq \left(\frac{(n-m)\gamma}{m}\right)^2$$

from which it follows that $m(n-1)-(n-m)\gamma^2 \geq 0$. If equality holds then the eigenvalues $\theta_1, \dots, \theta_m$ must all be equal. If $m < \gamma^2$ then this implies that

$$n \leq \frac{m(\gamma^2 - 1)}{\gamma^2 - m}.$$

We also obtain the following.

4.1 Lemma. *Let $\mathcal{L}$ be a set of n equiangular lines in $\mathbf{R}^m$, let $x_1, \dots, x_n$ be unit vectors spanning these lines with Gram matrix $I + \gamma^{-1}S$, where $\gamma > 0$. Then*

$$\gamma^2 \leq \frac{m(n-1)}{n-m}$$

and, if equality holds then S has exactly two distinct eigenvalues. □

A set of n equiangular lines such that S has only two eigenvalues is the same thing as a *regular two-graph* on n vertices. Note that an equiangular set of $\binom{m+1}{2}$ lines in $\mathbf{R}^m$ will give equality in this lemma. (But this only gives two examples.) The matrix S in the lemma is symmetric with zero diagonal and entries ± 1 off the diagonal. If D is diagonal matrix of the same order with diagonal entries ± 1 then DSD and S are similar and DSD still is symmetric with zero diagonal and entries ± 1 off the diagonal. We may choose D so that all off-diagonal entries of the first row and column of DSD are positive.

If S has only two eigenvalues then

$$S^2 + \alpha S + \beta I = 0 \tag{4.1}$$

for some α and β. Since $\operatorname{tr} S = 0$ and $\operatorname{tr} S^2 = n(n-1)$, taking the trace of Equation (4.1) yields that $\beta = n - 1$. If, as we may assume, S has the form

$$S = \begin{pmatrix} 0 & j^T \\ j^T & T \end{pmatrix}$$

then Equation (4.1) implies that

$$TJ = -\alpha J, \qquad T^2 + \alpha T - (n-1)I = -J. \tag{4.2}$$

From Equation (4.2) we deduce that $\frac{1}{2}(T + J - I)$ is the adjacency matrix of a strongly regular graph on $n-1$ vertices. Such a graph must have $k = 2c$ and, conversely, any strongly regular graph with $k = 2c$ on $n-1$ vertices determines a regular two-graph on n vertices.

Two surveys on regular two-graphs appear in [40]. We mention one question.

4.2 Problem. *Is there a regular two-graph on 76 or 96 vertices?*

5. Hamilton Cycles

We consider the existence of Hamilton cycles in vertex transitive graphs. Ignoring K_2, there are only four known vertex transitive graphs without Hamilton cycles. Two of these are the Petersen and Coxeter graphs. The Coxeter graph can be defined as follows. An *antiflag* in a projective plane is an ordered pair (p, ℓ), where p is a point and ℓ is a line such that $p \notin \ell$. The vertices of the Coxeter graph are the 28 antiflags from the projective plane of order two. Two such antiflags (p, ℓ) and (q, m) are adjacent if the set

$$\{p, q\} \cup \ell \cup m$$

contains all points of the plane. For more on the Coxeter graph, see Section 12.3 in [8].

The remaining two non-Hamiltonian vertex transitive graphs are obtained from the Petersen and Coxeter graphs by 'blowing up' each vertex to a triangle. (Formally we take the line graph of the subdivision graphs of the Petersen and Coxeter graphs. The *subdivision graph* $S(G)$ of G is obtained by installing one vertex in the middle of each edge of G.) The problem with the blowing-up process is that the graphs produced by applying it a second time are no longer vertex transitive, although they are still cubic and have no Hamilton cycle. For proofs that the Coxeter graph has no Hamilton cycle, see [6, 47].

5.1 Problem. *Are there any more connected vertex-transitive graphs without Hamilton cycles?*

Still ignoring K_2, the known non-Hamiltonian vertex-transitive graphs are not Cayley graphs. Thus we are lead to ask whether all connected Cayley graphs have Hamilton cycles. All known connected vertex-transitive graphs have Hamilton paths, and Lovász has conjectured that all connected vertex-transitive graphs have Hamilton paths. Witte [49] has proved that all Cayley graphs of p-groups have Hamilton cycles. For a survey of results on Hamilton cycles in Cayley graphs see, e.g., [50].

Babai [3] gives an ingenious proof that a connected vertex-transitive graph on n vertices must contain a cycle of length at least $\sqrt{3n}$; no better lower bound is known. Mohar [34] derives an algebraic technique for showing that certain graphs do not have Hamilton cycles. We present a simplified version of this for the Petersen graph.

Let P denote the Petersen graph and suppose that C is a cycle of length ten in it. Then the edges not in C form a perfect matching in P and the vertices in the line graph of P corresponding to the edges of C induce a cycle of length ten. Thus P has a Hamilton cycle if and only if there is an induced copy of C_{10} in $L(P)$. For any graph X

let $\theta_i(X)$ denote the i-th largest eigenvalue of the adjacency matrix of X. By interlacing [17: Theorem 5.4.1], we know that if Y is an induced subgraph of X then $\theta_i(Y) \leq \theta_i(X)$. Since $\theta_7(C_{10}) > \theta_7(L(P))$, the Petersen graph cannot have a Hamilton cycle.

The only problem with this argument is that there seems to be no other interesting case where it works. It fails on the remaining three vertex-transitive graphs with no Hamilton cycles. It would be very interesting to find a modification of this technique which could be used to show that Coxeter's graph has no Hamilton cycle.

6. The Matchings Polynomial

Let $p(X,k)$ denote the number of k-matchings in the graph X, i.e., the number of matchings with exactly k edges. If X has n vertices then its *matchings polynomial* of X is defined to be

$$\mu(X,x) = \sum_{k\geq 0} (-1)^k p(X,k) x^{n-2k}.$$

It is known that the zeros of $\mu(X,x)$ are all real [17: Corollary 6.1.2] and that, if X has a Hamilton path, they are all simple. This leads us to ask:

6.1 Problem. *Is there a connected vertex-transitive graph X such that $\mu(X,x)$ does not have only simple zeros?*

This question is discussed at some length in [16]. There a graph X is defined to be θ-critical if, for each vertex u of X, the multiplicity of θ as a zero of $\mu(X \setminus u, x)$ is less than its multiplicity as a zero of $\mu(X,x)$. All vertex transitive graphs are θ-critical, by a straightforward argument. Therefore we could solve Problem 6.1 by showing that if X were a connected θ-critical graph then θ must be a simple zero of $\mu(X,x)$. This is known to be true if $\theta = 0$ (Gallai, see [33: Section 3.1]) or if X is a tree (Neumaier [35]). For details, see [16].

7. Characterising Line Graphs

If X is a line graph then the least eigenvalue of its adjacency matrix $A(X)$ is at least -2. Cameron, Goethals, Seidel and Shult proved a converse to this, which we want to discuss.

First, some definitions. A root system is, more or less, a set of vectors in $\mathbf{R}^m$ which is invariant under reflection in the hyperplane orthogonal to any vector in it. Let $e_1, \dots, e_m$ be the standard basis in $\mathbf{R}^m$. Then the root system A_m is the set of vectors

$$e_i - e_j, \quad i \neq j$$

and the root system D_m is the set of vectors

$$e_i \pm e_j, \quad i \neq j.$$

We define E_8 to be D_8 together with all vectors in $\mathbf{R}^8$ with entries $\pm\frac{1}{2}$ and an even number of positive entries. It is not hard to see that a graph X is the line graph of a bipartite graph if and only if $A(X)+2I$ is the Gram matrix of a subset of A_m. We define X to be a *generalised line graph* is it is the Gram matrix of a subset of D_m. Every line graph lies in D_m. Cameron et al. proved that if X is a graph with least eigenvalue at least -2 then it is either a line graph, a generalised line graph or $A(X)+2I$ is the Gram matrix of subset of E_8. This extended earlier work, in particular of Alan Hoffman.

Now Hoffman [22] has also proved that a graph with least eigenvalue greater than

$$-1-\sqrt{2}$$

and sufficiently large minimum valency is a generalised line graph. (Here 'sufficiently large' is determined by Ramsey theory, which means that it is only finite in a fairly technical sense :-) .)

7.1 Problem. *Is there a classification of the graphs X with $\theta_{\min}(X) > -1-\sqrt{2}$, analogous to that of the graphs with least eigenvalue at least -2?*

Let $\theta_2(X)$ denote the second-largest eigenvalue of X. Then for the complement $\overline{X}$ of X we have

$$\theta_{\min}(\overline{X}) \leq -1-\theta_2(X).$$

(This follows because we obtain $A(\overline{X})$ by adding the matrix J, with rank one, to $-I-A(X)$.) Hence if $\theta_{\min}(\overline{X}) > -1-\sqrt{2}$ then $\theta_2(X) < \sqrt{2}$. This indicates that it should also be interesting to classify the graphs X such that $\theta_2(X) < \sqrt{2}$.

Woo and Neumaier [51] have recently proved that, if X is a graph with least eigenvalue greater than the smallest root of the polynomial x^3+2x^2-2x-2 (approximately -2.4812) and the minimum valency of X is large enough, then $\theta_{\min}(X) \geq -1-\sqrt{2}$ and X has a well-determined structure. This result is very interesting, but it still makes use of Ramsey theory and requires a lower bound on the minimum valency of X.

8. Shannon Capacity

The *strong product* $X * Y$ of the graphs X and Y has vertex set $V(X) \times V(Y)$, with (u,v) adjacent to (u',v') if and only if

(a) $u \sim u'$ and $v \sim v'$,
(b) $u = u'$ and $v \sim v'$, or
(c) $u \sim u'$ and $v = v'$.

We denote the strong product of n copies of X by $X^{(n)}$. Let $\alpha(X)$ denote the maximum number of vertices in an independent set in X. It is not hard to show that, for any graphs X and Y we have

$$\alpha(X * Y) \geq \alpha(X)\alpha(Y)$$

and from this it follows by Fekete's lemma (see Lemma 11.6 in [27]) that the limit

$$\lim_{n\to\infty} \alpha(X^{(n)})^{1/n}$$

always exists. We call it the *Shannon capacity* of X. This quantity is of some interest in coding theory, but for further information about this we refer the reader to [30] and the references given there.

Let $\omega(X)$ be the size of the largest clique in the graph X. We recall that X is perfect if, for any induced subgraph Y of X, the chromatic number of Y is equal to $\omega(Y)$. All bipartite graphs are perfect, and there are many other classes of perfect graphs, almost as many as there are graph theorists who have studied them. (For more information see, e.g., [31].) Shannon showed that if X is perfect then its Shannon capacity is equal to $\alpha(X)$. However, using the fact that C_5 is self-complementary, it is not hard to show that

$$\alpha(C_5 * C_5) = 5$$

and so the Shannon capacity of C_5 is at least $\sqrt{5}$, while $\alpha(C_5) = 2$. In 1979 Lovász [31] settled a long-standing open problem by proving that the Shannon capacity of C_5 is $\sqrt{5}$. His methods enabled the Shannon capacity of many other graphs to be determined, but the following question is still open.

8.1 Problem. *What is the Shannon capacity of C_7?*

9. Perfect Codes

The *ball of radius* m about a vertex v in a graph X is the set of all vertices in X at distance at most m from v. If C is a subset of $V(X)$, the *packing radius* of C is maximum integer e such that the balls of radius e about the vertices in C are pairwise disjoint. An e-*code* in X is a subset with packing radius at least e and an e-code is *perfect* if the balls of radius e about its vertices partition $V(X)$. The Hamming graph $H(n,q)$ has the set of all sequences of length n from $\{0,\dots,q-1\}$ as its vertices, with two sequences adjacent if they agree on all but one coordinate. If X is the Hamming graph, e-codes and perfect codes are precisely the e-codes and perfect codes of standard coding theory. If $e \geq 3$ and q is a prime power then there are only two perfect e-codes (see, e.g., [8: p. 355], one in $H(11,3)$ and the other in $H(23,2)$.

The Johnson graph $J(v,k)$ has all k-subsets of a fixed v-subset as its vertices, with two k-subsets adjacent if and only if they intersect in exactly $k-1$ elements. Two k-subsets are then at distance i if and only if they have exactly $k-i$ elements in common. The graphs $J(v,k)$ and $J(v,v-k)$ are isomorphic, so we will assume that $v \geq 2k$. When $v = 2k$ and k is odd, the pair formed by a given k-subset and it complement is a perfect code with e equal to $\lfloor k/2 \rfloor$. Delsarte [13: p. 55] raised the following question.

9.1 Problem. *Is there a perfect code with more than two vertices in a Johnson graph?*

The strongest result is due to Roos [37], who proved that if there is a perfect code in $J(v,k)$ with packing radius e then

$$v \leq \frac{2e+1}{e}(k-1).$$

Hammond [21] proved that there are no perfect codes in $J(2v+1,v)$ and $J(2v+2,v)$.

Perfect codes in other classes of distance regular graphs can also be very interesting. Perfect codes in the Hamming graphs $H(n,q)$ are part of classical coding theory—if $e \geq 3$ and q is a prime power then the only perfect codes are binary and ternary Golay codes. Chihara [11] has proved that most of the known families of distance regular graphs do not contain perfect codes. The Johnson graphs are one family of exceptions here, and the closely related odd graphs are another.

The odd graph $O(k+1)$ has the k-subsets of a $(2k+1)$-set as its vertices, with two k-subsets adjacent if and only if they are disjoint. (Thus it has the same vertex set as

$J(2k+1,k)$.) The lines of a Fano plane form a perfect 1-code in $O(4)$ and the blocks of the Witt design on 11 points forms a perfect 1-code in $O(6)$. No other examples are known. It not hard to show that there is a perfect 1-code in $O(m+1)$ if and only if there is a Steiner system with parameters

$$(m-1)\text{–}(2m+1,m,1).$$

Hence these codes will not be easy to find. Smith [43] has proved that there are no perfect 4-codes in the odd graphs. Perhaps it can be proved that there are no perfect e-codes in the odd graphs for e sufficiently large (ideally for $e \geq 2$).

Of course we should not forget that there may be interesting classes of codes which are not perfect. Completely regular codes (see Chapter 11 in [8]) provide one example.

10. p-Ranks

Let $H_v(k,\ell)$ be the 01-matrix with rows and columns respectively indexed by the k- and ℓ-subsets of a fixed v-set, and with ij-entry equal to one if and only if the i-th k-subset is contained in the j-th ℓ-subset. When $k \leq \ell \leq v-\ell$, the rows of $H_v(k,\ell)$ are linearly independent over the rationals. A surprisingly large number of the applications of linear algebra to combinatorics rest on this fact. (Some of these are presented in [18].) For the earliest proof of independence known to me, see [20]. More recently Richard Wilson [48] determined the rank of $H_v(k,\ell)$ over all finite fields. It is not clear what the combinatorial implications of this will be, but surely it will be useful in time.

There is a so-called q-analog of this problem. Consider the incidence structure formed by the k- and ℓ-subspaces of a v-dimensional vector space over the field with q elements (where a k-space is incident with the ℓ-spaces which contain it). Then it is natural to want to know the rank of this matrix. Over what field? There are three cases. Over the rationals this has been known at least since Kantor [25]. For primes not dividing q the answer appears in [15]; the result is in fact analogous to Wilson's for the rank of $H_v(k,\ell)$ in positive characteristic. The most interesting case is still open.

10.1 Problem. *What is the p-rank for the incidence matrix of k-spaces versus ℓ-spaces of a v-dimensional vector space over a field of order p^r?*

11. Homomorphisms

Let X and Y be graphs. A mapping f from $V(X)$ to $V(Y)$ is a *homomorphism* if $f(u)$ is adjacent to $f(v)$ in Y whenever u is adjacent to v in X. Since we do not allow vertices to be adjacent to themselves, f must map edges of X to edges of Y. If Y is a complete graph with r vertices then the homomorphisms from X into Y correspond to the proper colourings of X using at most r vertices. The *product* $X \times Y$ of the graphs X and Y has vertex set

$$V(X) \times V(Y)$$

and (u, v) is adjacent to (u', v') if and only if u is adjacent to u' and v is adjacent to v'. (This is the natural product in the category of graphs and homomorphisms, if it helps.)

S. Hedetniemi has made the following conjecture.

11.1 Conjecture (Hedetniemi). *For any two graphs X and Y*

$$\chi(X \times Y) = \min\{\chi(X), \chi(Y)\}.$$

When $n = 3$ we can verify the conjecture by showing that the product of two odd cycles contains an odd cycle. For $n = 4$, it was proved true by El-Zahar and Sauer in 1985 [14]. The remaining cases are still open, in fact we cannot exclude the possibility that $\chi(X \times Y)$ is less than 16 for some pair of graphs X and Y with arbitrarily large chromatic number! (See [36] for this and, for some recent work on this problem, with more references, see [38].)

The *Kneser graph* $K(v, k)$ has all k-subsets of a fixed v-set as its vertices, with two k-subsets adjacent if they are disjoint. So $K(v, 1)$ is the complete graph K_v and $K(2v+1, v)$ is the odd graph $O(v + 1)$. (In particular, $K(5, 2)$ is Petersen's graph.) The following question was raised by Pavol Hell.

11.2 Problem. *For which pairs (X, Y) of Kneser graphs is there a homomorphism from X to Y?*

Stahl [44: Section 2] proved that if $v > 2k$ there is a homomorphism from $K(v, k)$ to $K(v - 2, k - 1)$. Hence there is a homomorphism from $K(v, k)$ to K_{v-2k+2}, i.e., the chromatic number of $K(v, k)$ is at most $v - 2k + 2$. Lovász [29] proved that equality holds here, from which it follows that if $v - 2k > v' - 2k'$ then there is no homomorphism from $K(v, k)$ to $K(v', k')$. Further $K(v', k)$ is an induced subgraph of $K(v, k)$ whenever $v' \leq v$.

For any integer r there is an obvious homomorphism from $K(v,k)$ into $K(rv,rk)$, but none into $K(v',kr)$ when $v' < rv$. (See the corollary to Theorem 9 in Stahl [44].)

Let X be a graph and let M be the matrix whose columns are the characteristic vectors of the maximal independent subsets of $V(X)$. The value of the linear program

$$\min \mathbf{1}^T x$$
$$Mx \geq \mathbf{1}$$
$$x \geq 0$$

is the fractional chromatic number of X. We denote it by $\chi^*(X)$. Note that $\chi(X)$ is the value of the 01-integer program obtained from this LP (by requiring the entries of x to be 0 or 1), and that this is also the value obtained if we require that x be an integer vector. Perles observed that if there is homomorphism from X to Y then $\chi^*(X) \leq \chi^*(Y)$. It is not hard to show that if X is vertex transitive then $\chi^*(X) = |V(X)|/\alpha(X)$, whence $\chi^*(K(v,k)) = v/k$. Thus we conclude, for example, that there is no homomorphism from $K(8,3)$ into $K(11,4)$. Nothing else seems to be known about existence or non-existence of homomorphisms between Kneser graphs.

Dennis Stanton has raised the following problem. Let $K_q(v,k)$ be the graph whose vertices are k-subspaces of the n-dimensional vector space over $GF(q)$, with two subspaces adjacent if and only if their intersection is zero. What is the chromatic number of $K_q(v,k)$? Clearly we are only interested in the case where $n \geq 2k$. When $q = 1$ the graph $K_q(v,k)$ reduces to the Kneser graph $K(v,k)$.

Remark: I am indebted to Pavol Hell, who has had to explain much of the above material to me on more than one occasion. I hope I have it right by now.

12. Compact Graphs

Let G be a graph with adjacency matrix A and let Γ be the set of all permutation matrices which commute with A. (Thus Γ is isomorphic to $\mathrm{Aut}(G)$.) By $S(A)$ we denote the set of all doubly stochastic matrices which commute with A. Then $S(A)$ is is the set of all matrices X such that

$$XA = AX, \quad X\mathbf{1} = X^T\mathbf{1} = \mathbf{1}, \quad X \geq 0$$

and therefore it is a convex polytope. We call G *compact* if $S(A)$ is equal to the convex hull of Γ or, equivalently, if the extreme points of $S(A)$ are all permutation matrices. The following problem is raised implicitly by Tinhofer at the end of [46].

12.1 Problem. *Is there a good characterisation of compact graphs?*

Tinhofer [45, 46] has proved a number of results concerning compact graphs. In particular he has shown that trees and cycles are compact, and that the disjoint union of isomorphic compact graphs is compact. It is an easy observation that a compact regular graph must be vertex transitive. The converse to this is false—in [19] it is noted that the line graph of the complete graph K_n is not compact, at least when $n \geq 7$, and that the automorphism group of a compact regular graph is a multiplicity-free permutation group with rank equal to to the number of distinct eigenvalues of G. Schreck and Tinhofer [39] prove that a regular graph G on p vertices, p prime, is compact if and only if $\mathrm{Aut}(G)$ is isomorphic to the dihedral group of order $2p$. Using this it can be shown (see [19]) that there is a polynomial time algorithm for determining if a regular graph on a prime number of vertices is compact.

The graphs with $S(A) = \{I\}$ can be recognised in polynomial time [19]. The set $S(A)$ is a semigroup, and the convex hull of Γ is a sub-semigroup of it. In [19] it is shown that each equitable partition π of G determines an idempotent element X_π of $S(A)$.

12.2 Problem. *Is $S(A)$ generated (as a semigroup) by the convex hull of Γ and matrices X_π, where π is equitable?*

A compact graph G has the property that the cells of any equitable partition are the orbits of some group of automorphisms of G. It is not clear if the converse is true. If false then the answer to the previous problem is no.

13. Edge-difference Polynomials

Let G be a graph with vertex set $V = \{1, \dots, n\}$ and edge set E. Define the *edge-difference* polynomial p_G by

$$p_G(x_1, \dots, x_n) = \prod_{(i,j)\in E,\ i<j} (x_i - x_j).$$

The zero set of this polynomial is a set of $|E|$ hyperplanes through the origin in $\mathbf{R}^n$. The number of regions into which $\mathbf{R}^n$ is divided by these hyperplanes is equal to the absolute value of the chromatic polynomial of G, evaluated at -1. Note that p_G is actually a function on *oriented* graphs, but changing the orientation leads at worst to a change in sign. If we expand p_G as a sum of monomials then the number of terms in the result equals the number of orientations of G and each term has degree $|E|$.

Now let $U(n,k)$ denote the set of all vectors in $\mathbf{R}^n$ which have k entries equal, and let $V(n,k)$ denote the set of all vectors with at most $k-1$ distinct entries. We have the following obvious result.

13.1 Lemma. *The graph G has independence number less than k if and only if p_G is zero on $U(n,k)$. Further, the chromatic number of G is at least k if and only p_G is zero on $V(n,k)$.* □

This suggests that we should be able to obtain information about the independence and chromatic numbers of G by analysing p_G but, it seems fair to say, no great progress has been made in this direction yet.

Lovász [32] proves that the ideal of polynomials which vanish on $V(n,k)$ is generated by the polynomials p_H, where H is any graph on $V(G)$ consisting of a k-clique and $n-k$ isolated vertices. He also shows that $\chi(G) \geq k$ if and only if we can write p_G in the form

$$p_G = p_{H_1} + \cdots + p_{H_N},$$

where each H_i is a graph on $V(G)$ containing a k-clique. We mention one problem, raised in both [26] and [32].

13.2 Question. *Is there a sequence of graphs G_i such that the minimum possible number of terms in the above expansion increases exponentially?*

Analogous results holds for the independence number, see [26, 32, 28]. De Loera [28] shows that certain natural bases for the ideals of polynomials vanishing on $U(n,k)$ and $V(n,k)$ are Gröbner bases, which means that there is an effective algorithm for testing whether p_G lies in one of these ideals.

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