A Recurrence for Counting Graphical Partitions

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Abstract

In this paper, we give a recurrence to enumerate the set $G(n)$ of partitions of a positive even integer n which are the degree sequences of simple graphs. The recurrence gives rise to an algorithm to compute the number of elements of $G(n)$ in time $O(n^4)$ using space $O(n^3)$. This appears to be the first method for computing $|G(n)|$ in time bounded by a polynomial in n, and it has enabled us to tabulate $|G(n)|$ for even $n \leq 220$.

1 Introduction

A *partition* of a positive integer n is a sequence of positive integers $(\pi_1, \pi_2, \ldots, \pi_l)$ satisfying $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_l$ and $\pi_1 + \pi_2 + \ldots + \pi_l = n$. Let $P(n)$ denote the set of all partitions of n. $P(0)$ contains only the empty partition, λ. A partition $\pi \in P(n)$ is *graphical* if it is the degree sequence of some simple undirected graph. For example, $(5,4,4,3,3,1)$ is the degree sequence of the graph in Figure 1(a), but $(5,4,4,2,2,1)$ is not graphical. Clearly, graphical partitions exist only when n is even, since the sum of the degrees of the vertices of a graph is equal to twice the number of edges. Let $G(n)$ denote the set of graphical partitions of n. For convenience, we will call the empty partition graphical, so that $|G(0)| = 1$.

Several necessary and sufficient conditions to determine whether an integer sequence is graphical are surveyed in [SH]. Perhaps the best known is the following condition due to Erdös and Gallai [EG]:

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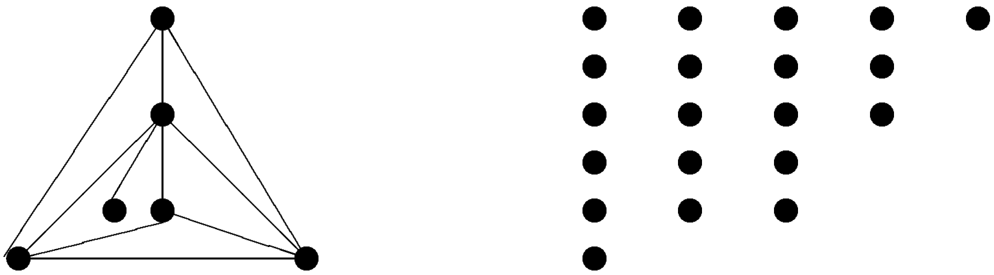

(a) (b)

Figure 1: (a) A graph with degree sequence $\pi = (5, 4, 4, 3, 3, 1)$ and (b) the Ferrars graph of π.

[Erdös - Gallai] A positive integer sequence $(\pi_1, \pi_2, \ldots, \pi_l)$, with $\pi_1 \geq \pi_2 \geq \ldots \geq \pi_l$, is graphical if and only if $\pi_1 + \pi_2 + \ldots + \pi_l$ is even and for $1 \leq j \leq l$,

$$\sum_{i=1}^{j} \pi_i \; \leq \; j(j-1) + \sum_{i=j+1}^{l} \min\{j, \pi_i\}.$$

In Section 2, we use a lesser-known condition to devise a recurrence to enumerate $G(n)$. As shown in Section 3, it can be used to count $G(n)$ in time $O(n^4)$ using space $O(n^3)$.

Our work was motivated by the following question, originally posed by Herbert Wilf, which remains open:

[Question] What fraction of the elements of $P(n)$ are graphic? In particular, does the ratio $|G(n)|/|P(n)|$ approach 0 as n approaches infinity?

To even plot the ratio $|G(n)|/|P(n)|$, it is necessary to compute $|G(n)|$, which, in our initial attempts, became a computational burden well before $n = 100$. Using an earlier version of the recurrence, we were able to compute $|G(n)|$ up to $n = 220$. These results are tabulated in Section 4. Where sufficient memory is available, computing $|G(n)|$ up to $n = 1000$ should be feasible.

For a related counting problem, we note that Stanley [St] has obtained a generating function for $f(n)$, the number of sequences $(x_1, x_2, \ldots, x_n)$ which are degree sequences of simple graphs with vertex set $\{v_1, v_2, \ldots, v_n\}$. Here, x_i is the degree of vertex v_i and the degree sequence is not necessarily nonincreasing.

2 The Recurrence

For a partition $\pi = (\pi_1, \ldots, \pi_l)$, the associated *Ferrars graph* is an array of l rows of dots, where row i has π_i dots and rows are left justified (Figure 1(b).) Let π' denote the conjugate partition $\pi' = (\pi'_1, \ldots, \pi'_m)$ where $m = \pi_1$ and π'_i is the number of dots in the i-th column of the Ferrars graph of π. The *Durfee square* of π is the largest square subarray of dots in the Ferrars graph of π. Let $d(\pi)$ denote the size (number of rows) of the Durfee square of π. The sequence

$$(\pi_1 - \pi'_1, \;\; \pi_2 - \pi'_2, \;\ldots,\; \pi_{d(\pi)} - \pi'_{d(\pi)})$$

is the sequence of *successive ranks* of π [At]. It will be convenient to work with the negatives of the ranks, so, for $1 \le i \le d(\pi)$, let $r_i(\pi) = \pi'_i - \pi_i$. We call $(r_1(\pi), \ldots r_{d(\pi)}(\pi))$ the sequence of successive *antiranks* of π.

The necessary and sufficient condition below, attributed to Nash-Williams, is proved in [RA2] and [SH].

[Nash-Williams] A partition π of an even integer is graphical if and only if for $1 \le j \le d(\pi)$,

$$\sum_{i=1}^{j} r_i(\pi) \ge j.$$

(This condition is called the *Hässelbarth Criterion* by the authors of [SH] since they first saw it in [Has], where it appeared without proof.) It can be shown that for $1 \le j \le d(\pi)$, the j-th Nash-Williams condition is equivalent to the j-th Erdös-Gallai condition. Furthermore, if conditions $1, 2, \ldots, d(\pi)$ of Erdös-Gallai are satisfied, then so are the remaining Erdös-Gallai conditions [RA2].

Let $P(n, k, l)$ be the set of partitions of n into at most l parts with largest part at most k and define $G(n, k, l)$ to be the set of graphical partitions in $P(n, k, l)$. Let $\pi \in P(n, k, l)$ and let α be obtained from π by deleting the first row and column of the Ferrars graph of π. Then $d(\alpha) = d(\pi) - 1$. Let $s = r_1(\pi) = \pi'_1 - \pi_1$. By the Nash-Williams condition, π is graphical if and only if $s > 0$ and for $1 \le j \le d(\alpha)$, the antiranks of α satisfy an s-variant of the Nash-Williams conditions:

$$s \;+\; \sum_{i=1}^{j} r_i(\alpha) \ge j.$$

With this in mind, define $P(n,k,l,s)$ for $s \geq 0$ by

$$P(n,k,l,s) = \{\ \pi \in P(n,k,l)\ \mid\ s\ +\ \sum_{i=1}^{j} r_i(\pi)\ \geq\ j\ \text{ for }\ 1 \leq j \leq d(\pi)\}.$$

Let $P(n,k,l,s) = \emptyset$ if $s < 0$ and note that for $s \geq 0$, $P(n,k,l,s) = \{\lambda\}$ if $P(n,k,l) = \{\lambda\}$.

Lemma 1 below is a restatement of the Nash-Williams condition and Lemma 2 follows since $G(n) = G(n,n,n)$.

Lemma 1 *For even $n \geq 0$, $G(n,k,l) = P(n,k,l,0)$.*

Lemma 2 *For even $n \geq 0$, $G(n) = P(n,n,n,0)$.*

Thus, we can compute $|G(n)|$ by computing $|P(n,k,l,s)|$ for appropriate values of the arguments. To this end, let $P'(n,k,l)$ and $P'(n,k,l,s)$, be the subsets of $P(n,k,l)$ and $P(n,k,l,s)$, respectively, consisting of those partitions into *exactly* l parts with largest part of size *exactly* k.

Lemma 3 *For $n > 0$ and $1 \leq k,l,s \leq n$,*

$$|P(n,k,l,s)| - |P'(n,k,l,s)| \ =\ |P(n,k-1,l,s)| + |P(n,k,l-1,s)| - |P(n,k-1,l-1,s)|.$$

Proof. ¿From the definitions of P and P' we have

$$P(n,k,l,s)\ \setminus\ P'(n,k,l,s)\ =\ P(n,k-1,l,s)\ \cup\ P(n,k,l-1,s).$$

The set on the left-hand side of this equality has size

$$|P(n,k,l,s)|\ -\ |P'(n,k,l,s)|$$

and by inclusion-exclusion, the set on the right-hand side of the equality has size

$$|P(n,k-1,l,s)|\ +\ |P(n,k,l-1,s)|\ -\ |P(n,k-1,l,s)\ \cap\ P(n,k,l-1,s)|.$$

The result follows since the intersection in the last term is $P(n,k-1,l-1,s)$. $\square$

Lemma 4 *Assume $n > 0$, $1 \leq k,l, \leq n$, and $s \geq 0$. Then*

$$|P'(n,k,l,s)|\ =\ |P(n-k-l+1,\quad k-1,\quad l-1,\quad s+l-k-1)|.$$

Proof. Define a function f on $P'(n,k,l)$ by $f(\pi) = \alpha$, where α is obtained from π by deleting the first row and column in the Ferrars graph of π. Given the assumptions of the theorem, if $P'(n,k,l) = \emptyset$ then either (1) $n < k+l-1$, in which case $n-k-l+1 < 0$ or (2) $n > kl$, which implies $n-k-l+1 > kl-k-l+1 = (k-1)(l-1)$. In either of these cases, $P(n-k-l+1, k-1, l-1) = \emptyset$. If $P'(n,k,l)$ contains only the partition $(k,1,\ldots,1)$, then $f((k,1,\ldots,1)) = \lambda$, $n-k-l+1 = 0$, and $P(n-k-l+1, k-1, l-1) = \{\lambda\}$. Otherwise, $d(\alpha) = d(\pi) - 1$ and $\alpha = (\alpha_1, \ldots \alpha_m)$ where $m = \pi'_2 - 1$, $\alpha_i = \pi_{i+1} - 1$ for $1 \le i \le m$ and $\alpha'_i = \pi'_{i+1} - 1$ for $1 \le i \le d(\pi) - 1$. Clearly, f is a bijection between $P'(n,k,l)$ and $P(n-k-l+1,\ k-1,\ l-1)$ Furthermore, for $1 \le j \le d(\pi)$,

$$\begin{aligned} s + \sum_{i=1}^{j} (\pi'_i - \pi_i) &= (s+l-k) + \sum_{i=2}^{j} (\pi'_i - \pi_i) \\ &= (s+l-k) + \sum_{i=2}^{j} ((\pi'_i - 1) - (\pi_i - 1)) \\ &= (s+l-k) + \sum_{i=1}^{j-1} (\alpha'_i - \alpha_i). \end{aligned}$$

Thus

$$s \ + \ \sum_{i=1}^{j} r_i(\pi) \ \ge \ j \iff$$

$$(s+l-k-1) \ + \ \sum_{i=1}^{j-1} r_i(\alpha) \ \ge \ j-1.$$

This establishes that $\pi \in P'(n,k,l,s) \iff \alpha \in P(n-k-l+1,\ k-1,\ l-1,\ s+l-k-1)$. $\square$

Lemma 5 *$P(n,k,l) = P(n,k,l,n) = P(n,k,l,s)$ for $s \ge n$.*

Proof. Note that for any $\pi \in P(n,k,l)$ and $1 \le j \le \pi(d)$,

$$\sum_{i=1}^{j} (\pi'_i - \pi_i - 1) \ \ge \ \sum_{i=1}^{j} -\pi_i \ \ge \ -n.$$

Thus, $\pi \in P(n,k,l,n)$, which means $P(n,k,l) \subseteq P(n,k,l,n)$. By definition, for $t' \ge t \ge 0$, $P(n,k,l,t) \subseteq P(n,k,l,t')$, thus for any $s \ge n$, $P(n,k,l,n) \subseteq P(n,k,l,s)$. The result follows since $P(n,k,l,s) \subseteq P(n,k,l)$. $\square$

The resulting recurrence for counting $|P(n,k,l,s)|$ is given in the following.

Theorem 1 *$|P(n,k,l,s)|$ is defined by:*

$$
\begin{array}{llll}
\multicolumn{4}{l}{|P(n,k,l,s)| =} \\
\quad \textit{if } ((n<0) \textit{ or } (k<0) \textit{ or } (l<0) \textit{ or } (s<0)) & \textit{then:} & 0 & (1) \\
\quad \textit{else if } n=0 & \textit{then:} & 1 & (2) \\
\quad \textit{else if } (k=0) \textit{ or } (l=0) & \textit{then:} & 0 & (3) \\
\quad \textit{else if } (k>n) & \textit{then:} & |P(n,n,l,s)| & (4) \\
\quad \textit{else if } (l>n) & \textit{then:} & |P(n,k,n,s)| & (5) \\
\quad \textit{else if } (s>n) & \textit{then:} & |P(n,k,l,n)| & (6) \\
\quad \textit{else: } |P(n,k-1,l,s)| + |P(n,k,l-1,s)| - |P(n,k-1,l-1,s)| & & & (7) \\
\qquad\qquad + |P(n-k-l+1,\ k-1,\ l-1,\ s+l-k-1)| & & &
\end{array}
$$

Proof. $P(n,k,l,s)$ was defined to be empty when $s<0$. For the remaining conditions in (1) through (5) the value of $|P(n,k,l,s)|$ is clear. Condition (6) follows from Lemma 5. For the general case (7), equate $|P'(n,k,l,s)|$ in Lemmas 3 and 4 and then solve for $|P(n,k,l,s)|$. □

3 The Algorithm

The recurrence of Theorem 1 for computing $|P(n,k,l,s)|$ has a straightforward implementation as a dynamic programming algorithm which fills a 4-dimensional table of entries $T[a,b,c,d] = |P(a,b,c,d)|$ where $0 \le a \le n$, $0 \le b \le k$, $0 \le c \le l$, and $0 \le d \le n$. The table is filled in any order which guarantees that when the time comes to fill entry $T[n',k',l',s']$, the required entries $T[n',k'-1,l',s']$, $T[n',k',l'-1,s']$, $T[n',k'-1,l'-1,s']$, and $T[n'-k'-l'+1,k'-1,l'-1,s'+l'-k'-1]$ have already been filled and can be read from the table. The table uses space $O(n^2kl)$ and only constant time is required to fill in each entry. In particular, computing $|G(n)| = |P(n,n,n,0)|$ takes time and space $O(n^4)$.

The space can be asymptotically improved as follows. For $0 \le c \le l$, let T_c be the 3-dimensional table of entries $T_c[a,b,d] = |P(a,b,c,d)|$ for $0 \le a \le n$, $0 \le b \le k$, and $0 \le d \le n$. Then $|P(n,k,l,s)|$ can be computed by computing successively the tables T_0, $T_1, \ldots T_l$. Note from the recurrence of Theorem 1 that computing entries in table T_c requires access only to values in table T_c or table T_{c-1}. Thus, in computing $|P(n,k,l,s)|$, no more than two 3-dimensional tables need to be stored at any given time, reducing the space required to $O(n^2k)$. Thus, computing $|G(n)|$ can be done in $O(n^4)$ time with $O(n^3)$ space.

4 Concluding Remarks

Even with this polynomial time algorithm, computing $|G(n)|$ for $n > 200$ quickly becomes impractical because of the huge space requirements. An additional burden on space is that $|G(n)|$ gets large quickly so that some method must be used to manipulate and allocate enough storage for these large numbers. The following strategy was suggested by the referee: Select small primes $p_1 < p_2 < \ldots < p_s$ so that $p_1 p_2 \ldots p_s > G(n)$. For $i = 1, \ldots, s$, use the recurrence of Theorem 1 to compute $G_i(n) = G(n) \bmod (p_i)$. Then by the Chinese Remainder Theorem, $G(n)$ can be recovered from $G_1(n), \ldots, G_s(n)$. If, for example, the primes can be represented with 8 bits, time $O(n^4)$ will be spent computing each of s tables, but the 3-dimensional tables now need store only 8-bit integers.

For those interested in the values $|G(n)|$, or in the ratio $|G(n)|/|P(n)|$ from the open question of Section 1, we include Tables 1 and 2. To the best of our knowledge, the values had previously been computed only through $n = 40$, as noted in [ER] in an acknowledgement to Ron Read. From the data, it seems reasonable to make the conjecture that for even $n \geq 18$, $|G(n)|/|P(n)|$ is monotone decreasing, but we are not aware of any proof of this. The best results known at this time are that

$$\overline{\lim}_{n\to\infty} \frac{\sqrt{n}\ |G(n)|}{|P(n)|} \ \geq\ \frac{\pi}{\sqrt{6}}$$

(so the ratio cannot go to 0 faster than $1/\sqrt{n}$ [ER]) and that [RA1]

$$\overline{\lim}_{n\to\infty} \frac{|G(n)|}{|P(n)|} \ \leq\ .25.$$

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n	$\|G(n)\|$	$\|P(n)\|$	$\|G(n)\|/\|P(n)\|$
2	1	2	0.500000
4	2	5	0.400000
6	5	11	0.454545
8	9	22	0.409091
10	17	42	0.404762
12	31	77	0.402597
14	54	135	0.400000
16	90	231	0.389610
18	151	385	0.392208
20	244	627	0.389155
22	387	1002	0.386228
24	607	1575	0.385397
26	933	2436	0.383005
28	1420	3718	0.381926
30	2136	5604	0.381156
32	3173	8349	0.380046
34	4657	12310	0.378310
36	6799	17977	0.378205
38	9803	26015	0.376821
40	14048	37338	0.376239
42	19956	53174	0.375296
44	28179	75175	0.374845
46	39467	105558	0.373889
48	54996	147273	0.373429
50	76104	204226	0.372646
52	104802	281589	0.372181
54	143481	386155	0.371563
56	195485	526823	0.371064
58	264941	715220	0.370433
60	357635	966467	0.370044
62	480408	1300156	0.369500
64	642723	1741630	0.369035
66	856398	2323520	0.368578
68	1136715	3087735	0.368139
70	1503172	4087968	0.367706
72	1980785	5392783	0.367303
74	2601057	7089500	0.366889
76	3404301	9289091	0.366484
78	4441779	12132164	0.366116
80	5777292	15796476	0.365733
82	7492373	20506255	0.365370
84	9688780	26543660	0.365013
86	12494653	34262962	0.364669
88	16069159	44108109	0.364313
90	20614755	56634173	0.363999
92	26377657	72533807	0.363660
94	33671320	92669720	0.363348
96	42878858	118114304	0.363028
98	54481054	150198136	0.362728
100	69065657	190569292	0.362418
102	87370195	241265379	0.362133
104	110287904	304801365	0.361835
106	138937246	384276336	0.361556
108	174675809	483502844	0.361272
110	219186741	607163746	0.361001

Table 1: Sizes of $G(n)$ and $P(n)$ and their ratio for $2 \leq n \leq 110$.

n	$\|G(n)\|$	$\|P(n)\|$	$\|G(n)\|/\|P(n)\|$
112	274512656	761002156	0.360725
114	343181668	952050665	0.360466
116	428244215	1188908248	0.360200
118	533464959	1482074143	0.359945
120	663394137	1844349560	0.359690
122	823598382	2291320912	0.359443
124	1020807584	2841940500	0.359194
126	1263243192	3519222692	0.358955
128	1560795436	4351078600	0.358715
130	1925513465	5371315400	0.358481
132	2371901882	6620830889	0.358248
134	2917523822	8149040695	0.358021
136	3583515700	10015581680	0.357794
138	4395408234	12292341831	0.357573
140	5383833857	15065878135	0.357353
142	6585699894	18440293320	0.357136
144	8045274746	22540654445	0.356923
146	9815656018	27517052599	0.356711
148	11960467332	33549419497	0.356503
150	14555902348	40853235313	0.356297
152	17692990183	49686288421	0.356094
154	21480510518	60356673280	0.355893
156	26048320019	73232243759	0.355695
158	31551087790	88751778802	0.355498
160	38173235010	107438159466	0.355304
162	46134037871	129913904637	0.355112
164	55694314567	156919475295	0.354923
166	67163674478	189334822579	0.354735
168	80909973315	228204732751	0.354550
170	97368672089	274768617130	0.354366
172	117056456152	330495499613	0.354185
174	140584220188	397125074750	0.354005
176	168675124141	476715857290	0.353827
178	202182888436	571701605655	0.353651
180	242116891036	684957390936	0.353477
182	289666252014	819876908323	0.353305
184	346234896845	980462880430	0.353134
186	413474657328	1171432692373	0.352965
188	493331835384	1398341745571	0.352700
190	588093594457	1667727404093	0.352632
192	700451190712	1987276856363	0.352468
194	833561537987	2366022741845	0.352305
196	991134281267	2814570987591	0.352144
198	1177516049387	3345365983698	0.351984
200	1397805210533	3972999029388	0.351826
202	1657968320899	4714566886083	0.351669
204	1964994991232	5590088317495	0.351514
206	2327052859551	6622987708040	0.351360
208	2753697110356	7840656226137	0.351207
210	3256081386335	9275102575355	0.351056
212	3847232865612	10963707205259	0.350757
214	4542341563460	12950095925895	0.350757
216	5359127512113	15285151248481	0.350610
218	6318223879596	18028182516671	0.350464
220	7443670977177	21248279009367	0.350319

Table 2: Sizes of $G(n)$ and $P(n)$ and their ratio for $112 \leq n \leq 220$.

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