Rigidity of integral curves of rank 2 distributions*

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1. Introduction

In this paper, we will investigate the geometry of "rigid" integral curves of rank 2 distributions on manifolds. Recently, the geometry of curves in a manifold M tangent to a specified distribution $\mathcal{D} \subset TM$ has been making a reappearance in differential geometry (for example, see Hamenstädt (1990), Pansu (1989), or Strichartz (1986,1989)). The study of these curves also has longstanding, close ties with control theory and the calculus of variations with "non-holonomic constraints". A natural approach to the study of these curves is to generalize the treatment of path spaces à la Morse theory and study the space $\Omega_{\mathcal{D}}(p, q)$ consisting of differentiable curves in M joining p to q and staying tangent to the distribution $\mathcal{D}$. At most of its points, the space $\Omega_{\mathcal{D}}(p, q)$, after being endowed with an appropriate topology, behaves very much like an infinite dimensional manifold. However, there are sometimes special curves $\gamma \in \Omega_{\mathcal{D}}(p, q)$ around which the local structure of $\Omega_{\mathcal{D}}(p, q)$ is drastically different. In this paper, we show that, for most distributions $\mathcal{D}$ of rank 2, such special curves always occur. (The precise meaning of "most" will be made clear in the following sections.)

The study of these special (or "non-regular") curves is a old subject, with early work having been done by Engel, Goursat, Cartan, Hilbert, and Bliss, while more modern work (in the context of differential geometry or the calculus of variations) has been done by Mūto, Gardner, and Hermann. The subject also has close ties with control theory and sub-Riemannian geometry, but we do not explore those in the present paper.

Instead, we are concerned with studying the curves γ at which $\Omega_{\mathcal{D}}(p, q)$ fails to be a smooth manifold when endowed with the natural C^1-topology. More precisely, we are interested in the so-called *non-regular* curves γ where a natural candidate for the tangent space $T_\gamma\Omega_{\mathcal{D}}(p, q)$ fails to be the true tangent space. In fact, this paper concentrates on the case of *rigid* $\mathcal{D}$-curves, that is, points $\gamma \in \Omega_{\mathcal{D}}(p, q)$ which are essentially isolated. In Sect. 2, we remind the reader of the notion of the natural tangent space $T_\gamma\Omega_{\mathcal{D}}(p, q)$ as well as the notion of non-regular curves in this context.

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That rigid curves exist is not new. In fact, examples of rigid curves were known for a certain rank 2 distribution $\mathcal{D}$ on $\mathbb{R}^4$ studied by Engel. What does appear to be new is the fact that rigid $\mathcal{D}$-curves are quite common for nearly all rank 2 distributions. In fact, our main result, Thm. 3.1, is that a rank 2 distribution $\mathcal{D}$ which satisfies some mild non-degeneracy conditions always has rigid $\mathcal{D}$-curves. Actually, our theorem is more precise than this; in the course of its proof we give a new local normal form for such distributions which allows one to check the rigidity of these non-regular $\mathcal{D}$-curves quite easily. This local normal form is likely to be useful in other contexts as well.

Mikhael Gromov (private communication) has pointed out to us that these examples of rigid $\mathcal{D}$-curves show that the sheaf of C^∞-immersions $\gamma: [a, b] \to M$ which are everywhere tangent to a bracket generating distribution $\mathcal{D}$ need not be micro-flexible in his sense, contrary to one's natural expectation (cf. Gromov (1986), p. 84).

In Sect. 3, we also consider the phenomenon of *local rigidity* and, when $\mathcal{D}$ is locally isomorphic to an Engel system, we give a necessary and sufficient condition for a locally rigid $\mathcal{D}$-curve γ to be globally rigid. This condition takes the form of determining whether the developing map of a certain canonical projective structure on γ has sufficiently large image in $\mathbb{RP}^1$. This test for global rigidity could quite probably be generalized to the case of rigid $\mathcal{D}$-curves where $\mathcal{D}$ is a system of Goursat type (defined in Sect. 4), but we do not do this in this paper.

In Sect. 4, we turn to some interesting examples. We point out that, for the generic rank 2 distribution $\mathcal{D}$ on a manifold M of dimension 5 or more (see Sect. 4 for the precise meaning of "generic"), there is at least one locally rigid $\mathcal{D}$-curve passing through each point $m \in M$ in each tangent direction in $\mathcal{D}_m$. When M has dimension exactly 5, the distributions which are generic in our sense are precisely the distributions studied by Cartan (1910). We show that, in this case, there is precisely one rigid $\mathcal{D}$-curve passing through each point in each $\mathcal{D}$-direction.

As an example of a system of Cartan type, we analyse the non-holonomic mechanical system which describes rolling one surface over another in space without slipping or twisting. In the case where one of the surfaces is a sphere, this is a well-known mechanical system. For example, when one surface is a sphere and the other is a plane, this system is mentioned in Arnold (1989) and in a recent preprint of Brockett and Dai. In the more general case of two arbitrary surfaces, we show that, if the surfaces have unequal Gaussian curvatures, then the rank 2 distribution which describes this mechanical system is of Cartan type and we interpret its rigid curves in terms of the geodesics on the two surfaces.

As a final example, we consider the distribution $\mathcal{D}$ on the orthonormal frame bundle $\mathcal{F}$ of $\mathbb{E}^3$ whose integral curves are the Frenet frames of curves of constant curvature $\kappa \equiv 1$. We show that there is a 7-parameter family of rigid $\mathcal{D}$-curves in this case. Most of the corresponding curves in space do not have constant torsion and hence are not generated by a 1-parameter subgroup of the Euclidean motion group. This shows that the rigid curves of a distribution may very well not be homogeneous even when the distribution itself is homogeneous.

Finally, we remark that the study of rigid curves for distributions of rank greater than 2 is more subtle. For example, it turns out that generic distributions of rank 3 in $\mathbb{R}^5$ or $\mathbb{R}^6$ do not have any rigid curves, even though they have non-regular ones. Thus, for distributions of greater rank, it appears that rigidity is a rarer phenomenon.

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2. Integral curves of distributions

We begin with some basic definitions. We are going to be interested in the geometry of curves (in manifolds) which are subject to what are often called *non-holonomic constraints* in the literature. In the language of exterior differential systems (Griffiths,1983), this is the geometry of 1-dimensional integral manifolds of a Pfaffian system.

Let M^{n+s} be a connected smooth manifold of dimension $n+s$ and let $\mathcal{D} \subset TM$ denote a subbundle of rank n on M. The manifold M will be our model of a (generalized) control system with n controls.

A smooth curve $\gamma: I \to M$ (where I is an interval or the circle) will be said to be an integral curve of $\mathcal{D}$ (or, more simply, a $\mathcal{D}$-curve) if $\gamma'(t)$ lies in $\mathcal{D}_{\gamma(t)}$ for all $t \in I$. Note that we do not require that γ be an immersion. However, to avoid obvious trivial cases, we shall henceforth assume that γ is *not* a constant map.

A distribution $\mathcal{D}$ on a manifold M is said to be *bracket generating* if, for every $v \in T_mM$, there exist some number k of vector fields $X_1, \ldots, X_k$ on M which are each everywhere tangent to the distribution $\mathcal{D}$ so that the iterated Lie bracket $Y = \big[X_1, [X_2, \cdots, [X_{k-1}, X_k] \cdots\big]$ has the value v at m. A distribution with the bracket generating property is often said to satisfy *Hörmander's condition*. A theorem of Chow (1939) asserts that if $\mathcal{D}$ is bracket generating then any two points of M can be joined by a $\mathcal{D}$-curve.[1] Let us assume from now on that $\mathcal{D}$ is bracket generating and let $\Omega_{\mathcal{D}}(p, q)$ denote the set of $\mathcal{D}$-curves $\gamma: [a, b] \to M$ which satisfy $\gamma(a) = p$ and $\gamma(b) = q$. We shall endow $\Omega_{\mathcal{D}}(p, q)$ with the C^1 topology.[2]

2.1 The local structure of the space of $\mathcal{D}$-curves

A fundamental problem is to describe the topology of $\Omega_{\mathcal{D}}(p, q)$. By Chow's theorem, we already know that $\Omega_{\mathcal{D}}(p, q)$ is non-empty. In fact, it is not hard to show that $\Omega_{\mathcal{D}}(p, q)$ has at least as many components as $\pi_1(M)$. However, less is known about the higher homotopy groups of $\Omega_{\mathcal{D}}(p, q)$ and, consequently, about things like the space of unstable critical points of various natural functionals on $\Omega_{\mathcal{D}}(p, q)$.

Naturally, one wants to regard $\Omega_{\mathcal{D}}(p, q)$ as an infinite dimensional manifold of some sort so as to apply Morse theory ideas. But it was recognized early on that this approach has problems caused by the presence of so-called "non-regular" curves. We now want to recall what these are and how they arise in the problem of studying the local structure of $\Omega_{\mathcal{D}}(p, q)$.

If $\Omega_{\mathcal{D}}(p, q)$ is to be a manifold, it must have a tangent space. What is the natural candidate? When $\mathcal{D} = TM$, the obvious model for a tangent space is the space $\mathcal{A}^0_0\big(T_\gamma\big)$

[1] Unfortunately, Chow's theorem does *not* say that any two points of M can be joined by some *immersed* $\mathcal{D}$-curve. Although this is presumably true, there does not seem to be a proof available.

[2] One should keep other topologies in mind as well. In fact, other topologies are often quite interesting. For example, in connection with sub-Riemannian geometry the Sobolev H^1 topology is important, see Hamenstädt (1990), Pansu (1989), and Strichartz (1986).

which consists of sections of $T_\gamma = \gamma^*\big(TM\big)$ which vanish at the endpoints. In other words, it is the space of tangent vector fields along γ which vanish at the endpoints. Of course, this works; it is the foundation of the classical methods of the calculus of variations.

In the case of $\mathcal{D}$-curves where $\mathcal{D}$ is a proper subbundle, care must be taken. For a given $\gamma \in \Omega_{\mathcal{D}}(p,q)$, set $T_\gamma = \gamma^*\big(TM\big)$ as before and set $Q_\gamma = \gamma^*\big(TM/\mathcal{D}\big)$. One first constructs a first order differential operator $D_\gamma\colon \mathcal{A}^0\big(T_\gamma\big) \to \mathcal{A}^1\big(Q_\gamma\big)$ with the property that $V \in \mathcal{A}^0\big(T_\gamma\big)$ is of the form

$$V(t) = \left(t,\, \frac{\partial \Gamma}{\partial s}(t,0)\right)$$

for some 1-parameter family of $\mathcal{D}$-curves $\Gamma(\cdot,s)\colon [a,b] \to M$ if and only if V satisfies $D_\gamma(V) = 0$. (In the coordinate formulation, this "variational operator" D_γ is quite classical. For an account with historical notes, see Bliss (1930).)

We shall have to compute a few examples in the following sections, so we will give a brief description of D_γ in a formulation suitable for those computations. Suppose that $\gamma\big([a,b]\big)$ lies in a region U on which the bundles $\mathcal{D}$ and TM are trivial. We may then choose linearly independent 1-forms $\omega^1,\dots,\omega^n,\theta^1,\dots,\theta^s$ on U with the property that, for all $m \in U$,

$$\mathcal{D}_m = \{v \in T_mM \mid \theta^\alpha(v) = 0\}\,.$$

(We shall often say that $\mathcal{D}$ is defined in U by the Pfaffian equations $\theta^1 = \cdots = \theta^s = 0$.) Associated to this coframing, there exist functions $C^\alpha_{ij} = -C^\alpha_{ji}$ (unique) and 1-forms ϕ^α_β (unique modulo the span of the θ^γ's) so that the following *structure equations* hold (note the use of the summation convention):

$$d\theta^\alpha = -\phi^\alpha_\beta \wedge \theta^\beta + \tfrac{1}{2} C^\alpha_{ij}\,\omega^i \wedge \omega^j \tag{2.1}$$

If $X_1,\dots,X_n,Y_1,\dots,Y_s$ is the basis of vector fields on U dual to $\omega^1,\dots,\omega^n,\theta^1,\dots,\theta^s$, then any $V \in \mathcal{A}^0\big(T_\gamma\big)$ can be written in the form

$$V(t) = \big(t,\ u^i(t)\,X_i\big(\gamma(t)\big) + v^\alpha(t)\,Y_\alpha\big(\gamma(t)\big)\ \big) \tag{2.2}$$

for some unique functions u^i and v^α on $[a,b]$. The formula for D_γ then takes the form

$$D_\gamma(V)(t) = \big(t,\ \overline{Y}_\alpha(t) \otimes \big(dv^\alpha + \phi^\alpha_\beta\big(\gamma'(t)\big)\,v^\beta\,dt + C^\alpha_{ij}\big(\gamma(t)\big)\omega^j\big(\gamma'(t)\big)\,u^i\,dt\big)\ \big)\,, \tag{2.3}$$

where $\overline{Y}_\alpha(t) \in T_{\gamma(t)}M/\mathcal{D}_{\gamma(t)}$ is the obvious reduced vector. (For a verification that this is indeed a correct formula for D_γ and that this operator is well-defined independent of the choice of coframing, see Hsu (1992).)

Now, the most obvious candidate for $T_\gamma\big(\Omega_{\mathcal{D}}(p,q)\big)$ is $\ker_0 D_\gamma = \ker D_\gamma \cap \mathcal{A}^0_0(T_\gamma)$, i.e., the "$\mathcal{D}$-variational vector fields along γ which vanish at the endpoints". However, unless $D_\gamma\big(\mathcal{A}^0_0(T_\gamma)\big) = \mathcal{A}^1(Q_\gamma)$, it can happen that some elements of $\ker_0 D_\gamma$ are not the first variation field of any smooth curve in $\Omega_{\mathcal{D}}(p,q)$. In fact, this can fail spectacularly. As Bliss (1930) points out, there are cases where the only elements of $\Omega_{\mathcal{D}}(p,q)$ in a C^1-open neighborhood of γ are reparametrizations of γ! We will see many such examples in Sects. 3 and 4 below.

On the other hand, it is an easy consequence of the Implicit Function Theorem (the classical finite dimensional one) that, if the map $D_\gamma\colon \mathcal{A}^0_0\big(T_\gamma\big) \longrightarrow \mathcal{A}^1\big(Q_\gamma\big)$ is surjective, then every element of $\ker_0 D_\gamma$ is, in fact, the first variation field of some curve in $\Omega_{\mathcal{D}}(p,q)$ which passes through γ. A proof which works in a local coordinate chart can be found in Bliss (1930, Sect. 7) and this argument is easily extended to the manifold case.

It has become standard to call a $\mathcal{D}$-curve $\gamma \in \Omega_{\mathcal{D}}(p,q)$ *normal* when the map D_γ is surjective. However, because of the way the Implicit Function Theorem is utilized in the analysis of such curves, we prefer the term *regular*, in analogy with the notion of a regular point of a smooth mapping.

In any case, the cokernel of D_γ, denoted by

$$\mathcal{H}_\gamma = \frac{\mathcal{A}^1(Q_\gamma)}{D_\gamma\left(\mathcal{A}^0_0(T_\gamma)\right)},$$

is always of finite dimension, and, in fact, has $\dim \mathcal{H}_\gamma \leq s$. Indeed, because the D_γ has surjective symbol, the usual closed range theorems show that its dual space $\mathcal{H}^*_\gamma$ is isomorphic to the kernel of the formal adjoint

$$D^*_\gamma\colon \mathcal{A}^0\left(Q^*_\gamma\right) \longrightarrow \mathcal{A}^1\left(T^*_\gamma\right).$$

Relative to a coframing described as above, we have the following formula for D^*_γ: Letting $\overline{\omega}^i$ and $\overline{\theta}^\alpha$, respectively, denote the obvious basis of sections of T^*_γ, then

$$\begin{aligned} D^*_\gamma\left(s_\alpha\,\overline{\theta}^\alpha\right)(t) = {}& \overline{\omega}^i(t) \otimes \left(C^\alpha_{ij}\left(\gamma(t)\right)\omega^j\left(\gamma'(t)\right)\,s_\alpha\,dt\right) \\ & + \overline{\theta}^\alpha(t) \otimes \left(-ds_\alpha + \phi^\beta_\alpha\left(\gamma'(t)\right)\,s_\beta\,dt\right)\ . \end{aligned} \tag{2.4}$$

Thus, a $\mathcal{D}$-curve γ is non-regular if and only if there exists a non-zero solution $p = (p_\alpha)$ to the system of equations

$$\begin{aligned} dp_\alpha &= \gamma^*\left(\phi^\beta_\alpha\right)\,p_\beta \\ 0 &= \gamma^*\left(C^\beta_{ij}\omega^j\right)\,p_\beta\ . \end{aligned} \tag{2.5}$$

Note that, because the first set of these equations is a determined set of linear ODE for the functions p_α, every solution $p = (p_\alpha)$ which is not identically zero is nowhere vanishing.[3]

This test for non-regularity can be formulated globally. First, recall that the cotangent bundle $\pi\colon T^*M \to M$ has a canonical symplectic structure $\Omega = d\omega$, where ω is the unique 1-form on T^*M with the property that $\omega(v) = \xi\left(\pi'(v)\right)$ for all $v \in T_\xi\left(T^*M\right)$. Using this symplectic structure, our test for non-regularity is codified in the following proposition, which, again, is simply a global rephrasing of the classical conditions for "normality". It will be used as our test for non-regularity in the rest of the paper.

Proposition 2.1 *Let $Q^* \subset T^*M$ be the annihilator of the distribution $\mathcal{D} \subset TM$. Let Ψ be the pullback of the canonical symplectic 2-form Ω on T^*M to the submanifold Q^*. Then a $\mathcal{D}$-curve $\gamma\colon [a,b] \to M$ is non-regular if and only if it has a lifting $\tilde{\gamma}\colon [a,b] \to Q^*$ which misses the zero section and which satisfies $\tilde{\gamma}'(t) \lrcorner \Psi = 0$ for all $t \in [a,b]$.*

Proof. By definition, any section $\sigma \in \mathcal{A}\left(Q^*_\gamma\right)$ is of the form $\sigma(t) = \left(t, \tilde{\gamma}(t)\right)$ where $\tilde{\gamma}\colon [a,b] \to Q^*$ is a lift of γ. Now, it is a straightforward (local) calculation that $D^*_\gamma(\sigma) = 0$ if and only if $\tilde{\gamma}'(t) \lrcorner \Psi = 0$ for all $t \in [a,b]$. For more details, see Hsu (1992). □

[3] Eqs. (2.5) are called the "adjoint equations" in the control literature, see Pontrjagin, et al (1962).

Of course, it follows from Prop. 2.1 that every sub-curve of a non-regular $\mathcal{D}$-curve is also non-regular and, obversely, every extension of a regular $\mathcal{D}$-curve is also regular.

Recall that, for any closed 2-form Φ on a manifold N, a *characteristic curve* of Φ (or, more simply, a *Φ-characteristic*) is an immersed curve $\psi\colon [a, b] \to N$ which satisfies $\psi'(t) \lrcorner \Phi = 0$ for all $t \in [a, b]$. In the present case, it is easy to see that a characteristic curve of Ψ on Q^* minus its zero section projects to M to be a non-constant $\mathcal{D}$-curve (which is, of course, non-regular). Thus, we have a way of generating all of the non-regular $\mathcal{D}$-curves: we simply find the characteristic curves of Ψ on the "punctured" bundle Q^*.

Now, any given non-regular curve $\gamma\colon [a, b] \to M$ has at least a 1-parameter family of distinct liftings to Q^* as a Ψ-characteristic curve. To see this, recall that, since Q^* is a vector bundle, it has a natural action of $\mathbb{R}^*$, given by scalar multiplication in the fibers. This action clearly scales Ψ and hence carries Ψ-characteristic curves to Ψ-characteristic curves with the same projection to M. We will say that two Ψ-characteristic curves which differ by an action of an element of $\mathbb{R}^*$ are *homothetic*. Obviously, for the purpose of studying the non-regular $\mathcal{D}$-curves, we may as well regard homothetic Ψ-characteristics as equivalent.

2.2 The prevalence of non-regular curves

It is an interesting question just how frequently one needs to deal with non-regular curves. For example, might there not be a simple, frequently satisfied condition on distributions $\mathcal{D}$ which forces all $\mathcal{D}$-curves to be regular? Alas, this is not the case. In fact, systems $\mathcal{D}$ which have no non-regular $\mathcal{D}$-curves are something of a rarity.

For example, if n is odd, then the manifold $Q^* \subset T^*M$ will have odd dimension, since its dimension is $2s+n$. Thus, Ψ cannot be non-degenerate on Q^*. It follows that there will be many Ψ-characteristic curves, most of them transverse to the fibers of $Q^* \to M$. The projections of these curves will, of course, be non-regular.

Even if $n = 2p$ for some integer p, it very often happens that the set $Z \subset Q^* \setminus \mathbf{0}$ where Ψ is not of full rank (i.e., where $\Psi^{s+p} = 0$) contains a non-empty hypersurface $H \subset Z$ which submerses onto M. (For example, this is easily seen to be true if p is odd.) For dimension reasons, this hypersurface H will contain many Ψ-characteristic curves which project to M to be non-regular curves.

In fact, it is very rare that Z is empty (which would, of course ensure that there are no Ψ-characteristic curves in $Q^* \setminus \mathbf{0}$ and hence no non-regular $\mathcal{D}$-curves). For example, it can easily be shown that Z is empty if and only if $\mathcal{D}$ is strongly bracket generating.[4]

In Rayner (1962, Appendix 2), it is shown that, for given values of n and s, a distribution $\mathcal{D}$ of rank n exists on some manifold M^{n+s} with the property that Z is empty if and only if there exist s everywhere linearly independent vector fields on the $(n-1)$-sphere. Thus, for example, n must always be even if s is positive, n must be divisible by 4 if $s > 1$, and n must be divisible by 8 if $s > 3$. Clearly, this is much too special to be a generally useful criterion (although it might occasionally be useful).

Thus, it appears that a good understanding of the space $\Omega_{\mathcal{D}}(p, q)$ will very likely entail understanding non-regular $\mathcal{D}$-curves in $\Omega_{\mathcal{D}}(p, q)$.

[4] This means that *every* local non-vanishing vector field X_1 tangent to $\mathcal{D}$ belongs to a local basis $\{X_1, X_2, \ldots, X_n\}$ for the sections of $\mathcal{D}$ so that the $(2n-1)$ vector fields $X_1, \ldots, X_n, [X_1, X_2], \ldots, [X_1, X_n]$ span the local tangent vector fields on M. Of course, this clearly cannot hold unless $s \le n-1$, which is already a severe limitation on the usefulness of the criterion.

3. Rigidity in systems with $n = 2$

In this section, we are going to show that, not only are non-regular curves frequently encountered in the study of non-holonomic systems, but, in the first non-trivial case, an extreme form of non-regularity occurs, which we shall call *rigidity*. First, we make the following definition.

Definition 3.1 *A $\mathcal{D}$-curve $\gamma\colon [a,b] \to M$ is* rigid *if there is a C^1-neighborhood $\mathcal{U}$ of γ in $\Omega_{\mathcal{D}}\bigl(\gamma(a), \gamma(b)\bigr)$ so that every $\gamma_1 \in \mathcal{U}$ is a reparametrization of γ. We say that γ is* locally rigid *if every point of $I = [a,b]$ lies in a subinterval $J \subset I$ so that γ restricted to J is rigid.*

Note that rigid curves are at the opposite extreme from regular ones. In some sense, they are as badly behaved as possible. Nevertheless, we are going to show that, when $n = 2$ and $s > 1$ (the first non-trivial case) they are quite common.[5]

3.1 Engel systems

Consider the case where $n = 2$ and $s = 2$. The bracket generating assumption on the system $\mathcal{D} \subset TM$ ensures that, at least on an open subset of M, the system $[\mathcal{D},\mathcal{D}]$ has dimension 3 and the system $\bigl[[\mathcal{D},\mathcal{D}],\mathcal{D}\bigr]$ has dimension 4. We shall say that $\mathcal{D}$ is an *Engel system* on M if these dimension counts hold at every point of M. (We have chosen the name "Engel" because of the extensive work that Engel did on systems of this kind in connection with the theory of Monge characteristics.)

It was proved by Engel himself that these systems have a simple local normal form. In fact, according to Engel, each point of M has an open neighborhood U on which there exist local coordinates (w,x,y,z) in which the system $\mathcal{D}$ is simply the span of the vector fields

$$X_1 = \frac{\partial}{\partial x} + z\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z} \qquad \text{and} \qquad X_2 = \frac{\partial}{\partial w}. \tag{3.1}$$

The corresponding annihilator bundle $Q^*U \subset T^*U$ is then spanned by the 1-forms $\theta^1 = dy - z\,dx$ and $\theta^2 = dz - w\,dx$. We shall call such a local coordinate chart an *Engel chart* for $\mathcal{D}$. (For a proof of this normal form, see Bryant, et al (1991, Thm. II.5.1).)

Let us use the method suggested by Prop. 2.1 to compute the non-regular $\mathcal{D}$-curves which lie in the domain U of an Engel chart. It is clear that there is a diffeomorphism $Q^*U = U \times \mathbb{R}^2$, where we use coordinates p_1 and p_2 on the $\mathbb{R}^2$-factor, so that the 2-form Ψ is given by

$$\begin{aligned} \Psi &= d\bigl(p_1\,\theta^1 + p_2\,\theta^2\bigr) \\ &= dp_1 \wedge \theta^1 + (dp_2 + p_1\,dx) \wedge \theta^2 - p_2\,dw \wedge dx \end{aligned} \tag{3.2}$$

Clearly Ψ has full rank away from the locus $p_2 = 0$. Hence all of the characteristic curves of Ψ must lie in this hypersurface. Since we are only concerned with the curves which also satisfy $p_1 \neq 0$, it follows immediately that the characteristic curves of Ψ are defined by the Pfaffian equations

5 When $s = 1$ things are different: An everywhere non-integrable plane field $\mathcal{D}$ on a 3-manifold M is just a contact structure on M and every $\mathcal{D}$-curve is easily seen to be regular in this case. However, note that the "generic" 2-plane field $\mathcal{D}$ on M^3 will be a contact distribution only away from a certain closed hypersurface $H \subset M$ and that this hypersurface *will* contain non-regular curves, see Montgomery (1993).

$$\theta^1 = \theta^2 = dx = dp_1 = 0 \tag{3.3}$$

in the set where $p_2 = 0$ and $p_1 \neq 0$. Since the linear span of $\{\theta^1, \theta^2, dx\}$ is the same as that of $\{dx, dy, dz\}$, it follows that the non-regular $\mathcal{D}$-curves in U are the $\mathcal{D}$-curves on which x, y, and z are constant. Thus, U is foliated by a 3-dimensional family of non-regular $\mathcal{D}$ curves.

This calculation clearly globalizes, so that M itself has a canonical foliation whose leaves are precisely the immersed non-regular $\mathcal{D}$-curves. We summarize this in the following proposition.

Proposition 3.1 *Let a 4-manifold M^4 be endowed with an Engel system $\mathcal{D} \subset TM$. Then there is a canonical associated foliation $\mathcal{F}$ of M by curves which has the property that a $\mathcal{D}$-curve $\gamma\colon [a, b] \to M$ is non-regular if and only if the image of γ lies in a single leaf of $\mathcal{F}$.*

We are now going to show that, for an Engel system $\mathcal{D}$, the immersed non-regular $\mathcal{D}$-curves are locally rigid. First, we prove the following proposition.

Proposition 3.2 *Let $M = \mathbb{R}^4$ have coordinates (w, x, y, z) and let $\mathcal{D}$ be the Engel system spanned by the two vector fields*

$$X = \frac{\partial}{\partial x} + z\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z} \qquad \text{and} \qquad W = \frac{\partial}{\partial w}.$$

Then up to reparametrization, there is a unique $\mathcal{D}$-curve γ in $\mathbb{R}^4$ joining $p = (w_0, x_0, y_0, z_0)$ to $q = (w_1, x_0, y_0, z_0)$ and satisfying the condition that $\gamma^*(dw)$ is nowhere zero.

Proof . First, note that $\mathcal{D}$ is invariant under transformations of either of the forms

$$(w, x, y, z) \mapsto (w,\ x + x_0,\ y,\ z)$$

or

$$(w, x, y, z) \mapsto (w + w_0,\ x,\ y + y_0 + z_0\,x + \tfrac{1}{2} w_0\,x^2,\ z + z_0 + w_0\,x)$$

and that such transformations leave dw invariant. Next, note that $\mathcal{D}$ is also invariant under transformations of the form $(w, x, y, z) \mapsto (rw, x, ry, rz)$ where $r \neq 0$ is any constant and that these transformations merely replace dw by $r\,dw$. It follows from these observations that we may, without loss of generality, reduce to the case where $p = (0, 0, 0, 0)$ and $q = (1, 0, 0, 0)$.

Suppose now that $\gamma\colon [0, 1] \to \mathbb{R}^4$ is a $\mathcal{D}$-curve satisfying the conditions of the proposition and satisfying $\gamma(0) = p$ and $\gamma(1) = q$. Then the w-component of γ is clearly an increasing smooth function of t with non-vanishing derivative which maps $[0, 1]$ diffeomorphically onto itself.

Thus, γ can be reparametrized so as to be of the form $\gamma(t) = \big(t, x(t), y(t), z(t)\big)$. By construction, the functions $x(t)$, $y(t)$, and $z(t)$ are smooth functions on $[0, 1]$ which vanish at the endpoints and satisfy the differential equations

$$\begin{aligned} y'(t) &= z(t)\,x'(t)\ , \\ z'(t) &= t\,x'(t)\ . \end{aligned} \tag{3.4}$$

Since $x(0) = z(0) = 0$, it follows that

$$z(t) = t\,x(t) - \int_0^t x(\tau)\,d\tau\ , \tag{3.5}$$

from which it further follows that $x(t)$ is the derivative of a function $h(t)$ which also vanishes at the endpoints. This yields the formulae

$$x(t) = h'(t) \qquad \text{and} \qquad z(t) = t\,h'(t) - h(t)\ . \tag{3.6}$$

The differential equation for y now becomes

$$y'(t) = z(t)\,x'(t) = \big(t\,h'(t) - h(t)\big)\,h''(t)\ . \tag{3.7}$$

Since, by hypothesis, $y(0) = h(0) = 0$, integrating by parts gives

$$y(t) = \tfrac{1}{2}t\,\big(h'(t)\big)^2 - h(t)\,h'(t) + \frac{1}{2}\int_0^t \big(h'(\tau)\big)^2\,d\tau\ . \tag{3.8}$$

However, now setting $t = 1$ in this formula and using $h(1) = y(1) = 0$ gives

$$0 = \int_0^1 \big(h'(\tau)\big)^2\,d\tau\ . \tag{3.9}$$

It follows that $h'(t) \equiv 0$, so $h(t) \equiv 0$. Thus $\gamma(t) = (t, 0, 0, 0)$, as desired. □

This result has the following immediate corollary.

Proposition 3.3 *For any Engel structure $\mathcal{D}$ on a 4-manifold M, the leaves of the associated foliation $\mathcal{F}$ of non-regular $\mathcal{D}$-curves are locally rigid.*

Naturally, this raises the question of whether these locally rigid curves are globally rigid too. The following Proposition gives a necessary and sufficient condition for this by a criterion reminiscent of the theory of focal points in Riemannian geometry.

Proposition 3.4 *Let $\mathcal{D}$ be an Engel structure on a 4-manifold M and let $\mathcal{F}$ be the associated foliation by non-regular $\mathcal{D}$-curves. Then each leaf L of $\mathcal{F}$ carries a canonical projective structure. Moreover, an immersion $\gamma\colon [a,b] \to L$ is a rigid $\mathcal{D}$-curve if and only if some (and hence every) developing map $\delta_\gamma\colon [a,b] \to \mathbb{RP}^1$ of the induced projective structure on $[a,b]$ is one-to-one except possibly at the endpoints.*

Proof. First, we explain how the canonical projective structure on the leaves of $\mathcal{F}$ is constructed. Suppose that $p \in M$ is fixed and let (w, x, y, z) and (W, X, Y, Z) be any two Engel charts for $\mathcal{D}$ with domain U containing p. Thus, the Engel system $\mathcal{D}$ can be described by either of the following pairs of Pfaffian equations:

$$dy - z\,dx = dz - w\,dx = 0\ , \quad \text{or} \quad dY - Z\,dX = dZ - W\,dX = 0\ . \tag{3.10}$$

As we computed above, the leaves of the foliation $\mathcal{F}$ in U are given by $dx = dy = dz = 0$ and hence they must also be given by $dX = dY = dZ = 0$. It follows that the 1-forms dX, dY, and dZ are linear combinations of dx, dy, and dz. We may thus regard X, Y, and Z as functions of x, y, and z. Moreover, since direct calculation yields that $[\mathcal{D}, \mathcal{D}]$ is described by either $dy - z\,dx = 0$ or $dY - Z\,dX = 0$, we see that there must be a function λ so that $dY - Z\,dX = \lambda\,(dy - z\,dx)$ and another function μ so that

$$dZ - W\,dX \equiv \mu\,(dz - w\,dx) \mod (dy - z\,dx)\ . \tag{3.11}$$

Expanding this out and comparing coefficients yields the relation

$$w = \frac{(X_x + zX_y)\,W - Z_x - zZ_y}{-X_z\,W + Z_z}\;. \tag{3.12}$$

When restricted to each leaf L of $\mathcal{F}$, all of the functions X, Y, Z, and their partials with respect to x, y, and z become constant. Thus, Eq. (3.12) shows that, on each leaf L, the function w is well-defined up to a linear fractional transformation with constant coefficients. It follows that there exists a unique projective structure on each leaf L of $\mathcal{F}$ with the property that the first coordinate w of any Engel chart (w, x, y, z) is a projective coordinate on each leaf. This is the canonical projective structure whose existence was asserted.

Now, we want to see how this projective structure detects rigidity. Let $\gamma\colon [a, b] \to L$ be an immersion. Without loss of generality, we will assume that $[a, b] = [0, 1]$. Let $p = \gamma(0)$ and let $D^3 \subset \mathbb{R}^3$ be a disk centered on $\mathbf{0} \in \mathbb{R}^3$ and let $\phi\colon D^3 \to M$ be a smooth immersion with $\phi(\mathbf{0}) = p$ which is also transverse to L at p. By shrinking D if necessary, we may assume that $\phi(D)$ is transverse to $\mathcal{F}$ at all of its points. Since $\mathcal{F}$ is a foliation and $[0, 1]$ is compact, we may use the usual techniques to construct a smooth immersion $\Phi\colon D \times (-\varepsilon, 1+\varepsilon) \to M$ with the properties that first, $\Phi(\mathbf{0}, t) = \gamma(t)$ for all $t \in [0, 1]$; and, second, for each $\mathbf{u} \in D$, the curve $\gamma_{\mathbf{u}}(t) = \Phi(\mathbf{u}, t)$ is an immersion of $(-\varepsilon, 1+\varepsilon)$ into a leaf of $\mathcal{F}$.

Clearly, γ is rigid as a $\mathcal{D}$-curve if and only if the curve $\{\mathbf{0}\} \times [0, 1]$ is rigid in $D \times (-\varepsilon, 1+\varepsilon)$ as a $\Phi^*(\mathcal{D})$-curve. Thus, for the rest of the proof we may (and shall) assume that $M = D \times (-\varepsilon, 1+\varepsilon)$ and that Φ is simply the identity mapping.

Now consider the distribution $\mathcal{D}_1 = [\mathcal{D}, \mathcal{D}]$, which has rank 3. As noted above, in a Engel chart, this distribution is described by the Pfaffian equation $dy - z\,dx = 0$, which is clearly constant along the leaves of $\mathcal{F}$. It follows that, by shrinking D again if necessary, we may assume that there are independent functions x, y, and z, globally defined on $D \times (-\varepsilon, 1+\varepsilon)$ and vanishing at $(\mathbf{0}, 0)$, for which the Pfaffian equation $dy - z\,dx = 0$ describes the system $\mathcal{D}_1$ and moreover, so that the foliation $\mathcal{F}$ is described by the Pfaffian equations $dx = dy = dz = 0$.

Since $D \times (-\varepsilon, 1+\varepsilon)$ is simply connected, there exists, on $D \times (-\varepsilon, 1+\varepsilon)$, a function θ, unique up to additive multiples of π, so that the Engel system $\mathcal{D}$ is described by the Pfaffian equations

$$dy - z\,dx = \cos\theta\,dz - \sin\theta\,dx = 0\;. \tag{3.13}$$

The fact that these equations describe an Engel system implies that the functions (x, y, z, θ) are independent at every point of $D \times (-\varepsilon, 1+\varepsilon)$. Hence they define an embedding of $D \times (-\varepsilon, 1+\varepsilon)$ into $\mathbb{R}^4$.

We may thus now regard our problem as one of determining the rigidity of the non-regular $\mathcal{D}$-curves in $\mathbb{R}^4$ endowed with the Engel structure $\mathcal{D}$ defined by the (global) equations

$$dy - z\,dx = \cos\theta\,dz - \sin\theta\,dx = 0\;. \tag{3.14}$$

Note that, by construction, the map $\delta\colon D \times (-\varepsilon, 1+\varepsilon) \to \mathbb{RP}^1$ defined by

$$\delta(p) = \big[\cos\theta(p),\ \sin\theta(p)\big] \tag{3.15}$$

projectively develops each fiber of $\mathcal{F}$ into $\mathbb{RP}^1$ according to its canonical projective structure. Moreover, the developing map $\delta_\gamma(t) = \big[\cos\theta(\mathbf{0}, t),\ \sin\theta(\mathbf{0}, t)\big]$ is one-to-one on the open interval $(0, 1)$ if and only if $|\theta(\mathbf{0}, 1) - \theta(\mathbf{0}, 0)| \le \pi$.

In order to ease the following argument notationally, it is convenient to make a slight change of coordinates. If we replace y by $(y + xz)/2$, the Pfaffian equations describing $\mathcal{D}$ take the more symmetric form

$$dy - z\,dx + x\,dz = \cos\theta\,dz - \sin\theta\,dx = 0\ . \tag{3.16}$$

Also, let us note that we are free to apply coordinate changes of the following form: If a, b, c, and e are constants satisfying $ae - bc \neq 0$, then the equations

$$\begin{aligned} \tilde{x} &= a\,x + b\,z \\ \tilde{y} &= (ae - bc)\,y \\ \tilde{z} &= c\,x + e\,z \\ [\cos\tilde{\theta}, \sin\tilde{\theta}] &= \big[a\ \cos\theta + b\ \sin\theta, c\ \cos\theta + e\ \sin\theta\big] \end{aligned} \tag{3.17}$$

define a new set of coordinates (with $\tilde{\theta}$ globally defined uniquely up to an additive integral multiple of π) in which the defining equations of the distribution $\mathcal{D}$ are still of the form

$$d\tilde{y} - \tilde{z}\,d\tilde{x} + \tilde{x}\,d\tilde{z} = \cos\tilde{\theta}\,d\tilde{z} - \sin\tilde{\theta}\,d\tilde{x} = 0\ . \tag{3.18}$$

Thus, the mapping $(x, y, z, \theta) \mapsto (\tilde{x}, \tilde{y}, \tilde{z}, \tilde{\theta})$ is a diffeomorphism of $\mathbb{R}^4$ onto itself which preserves $\mathcal{D}$. This allows us to make a projective change of parameter in θ. Note that all changes of this form preserve the length of a θ-interval if and only if this length is an integer multiple of π.

By aid of such transformations, we are reduced to deciding which curves of the form $(x, y, z, \theta) = (0, 0, 0, t)$ with $0 \leq t \leq P$ are rigid. We claim that this $\mathcal{D}$-segment is rigid if and only if $P \leq \pi$. By the above remarks on the developing map δ, this claim is equivalent to the remaining part of the proposition to be proved.

First, suppose that $P \leq \pi$. Clearly this segment has a C^1-neighborhood $\mathcal{U}$ in $\Omega_{\mathcal{D}}\big((\mathbf{0}, 0), (\mathbf{0}, P)\big)$ which contains only curves on which the form $d\theta$ is non-vanishing. It follows that any curve $\tilde{\gamma}$ in $\mathcal{U}$ can be written in the form $\tilde{\gamma}(\theta) = \big(x(\theta), y(\theta), z(\theta), \theta\big)$ for some smooth functions $x(\theta)$, $y(\theta)$, and $z(\theta)$ on the interval $[0, P]$ which vanish at the endpoints $\theta = 0$ and $\theta = P$. Since we must have

$$\begin{aligned} 0 &= \cos\theta\,dz - \sin\theta\,dx \\ &= d(z\ \cos\theta - x\ \sin\theta) + (z\ \sin\theta + x\ \cos\theta)\,d\theta \end{aligned} \tag{3.19}$$

it follows that there must be a smooth function $h(\theta)$ defined on $[0, P]$ which satisfies

$$\begin{aligned} z\ \cos\theta - x\ \sin\theta &= \quad h \\ z\ \sin\theta + x\ \cos\theta &= -\,h' \end{aligned} \tag{3.20}$$

where, clearly, h and h' must vanish at the endpoints. We can solve these equations for x and z, obtaining

$$\begin{aligned} z(\theta) &= \quad h(\theta)\ \cos\theta - h'(\theta)\sin\theta \\ x(\theta) &= -h(\theta)\ \sin\theta - h'(\theta)\cos\theta \end{aligned} \tag{3.21}$$

and then substitute this into the relation $dy = z\,dx - x\,dz$ and finally integrate (using the endpoint conditions) to obtain

$$y(\theta) = h(\theta)h'(\theta) + \int_0^\theta \big(h(\xi)\big)^2 - \big(h'(\xi)\big)^2\,d\xi\ . \tag{3.22}$$

However, since $h(P) = y(P) = 0$, this gives

$$0 = \int_0^P \big(h(\xi)\big)^2 - \big(h'(\xi)\big)^2 \, d\xi \; . \tag{3.23}$$

Now, it is well known that for any non-zero differentiable function h on $[0, \pi]$ which vanishes at the endpoints, we have

$$\int_0^\pi \big(h(\xi)\big)^2 \, d\xi \le \int_0^\pi \big(h'(\xi)\big)^2 \, d\xi \; , \tag{3.24}$$

with equality if and only if h is a constant multiple of $\sin\theta$. Since we must also have $h'(0) = 0$, it follows that the only possibility for equality is $h \equiv 0$. Of course, this establishes that the given segment is rigid when $P \le \pi$.

Finally, to establish the non-rigidity for $P > \pi$, it suffices to exhibit an appropriate family of functions h_λ which describe a non-trivial deformation of the initial curve. As an example with $P = 2\pi/\sqrt{3} < 2\pi$, consider the 1-parameter family of functions

$$h_\lambda(\theta) = \lambda \left(1 - \cos\big(\sqrt{3}\,\theta\big) \right) \tag{3.25}$$

(where λ is a parameter). These all satisfy $h_\lambda(0) = h_\lambda(P) = h'_\lambda(0) = h'_\lambda(P) = 0$ and moreover the functions $x_\lambda(\theta)$, $y_\lambda(\theta)$ and $z_\lambda(\theta)$ constructed from h_λ by the above formulae all vanish at the endpoints. Thus, this segment is not rigid. Any longer interval cannot be rigid since we can merely extend this family of h's by zero past P. For shorter intervals (but still longer than π), we take advantage of the fact that the transformation group preserving $\mathcal{D}$ on $\mathbb{R}^4$ described above can be used to make equivalent any two intervals whose θ-length is strictly between π and 2π.

3.2 Rigidity in higher dimensional cases

We now turn to the higher dimensional case. We are going to show that if $\mathcal{D}$ is a non-integrable rank 2 distribution on M^{2+s} satisfying the condition that the rank 3 distribution $\mathcal{D}_1 = [\mathcal{D}, \mathcal{D}]$ also be non-integrable, then there always exist many non-regular $\mathcal{D}$-curves which are locally rigid.

First, we do some preliminary work on the structure equations. We may as well assume that $s > 2$, since the case $s = 2$ is just that of Engel systems. Let $\mathcal{D}$ be a non-integrable rank 2 distribution on M^{2+s} and let $U \subset M$ be an open set on which both $\mathcal{D}$ and TM are trivial bundles. Then, on U, there exist linearly independent 1-forms $\omega^1, \omega^2, \theta^1, \dots, \theta^s$ so that $\mathcal{D}$ is defined in U by the Pfaffian equations

$$\theta^1 = \theta^2 = \cdots = \theta^s = 0 \; . \tag{3.26}$$

As we have seen, there exist functions C^α on U so that

$$d\theta^\alpha \equiv C^\alpha \, \omega^1 \wedge \omega^2 \bmod \theta^1, \dots, \theta^s \; . \tag{3.27}$$

By hypothesis, these functions C^α do not vanish simultaneously at any point of U so it is possible to make a basis change in the θ^α so that we have the equations

$$\left.\begin{aligned} d\theta^1 \equiv \cdots \equiv d\theta^{s-1} &\equiv \; 0 \\ d\theta^s &\equiv \omega^1 \wedge \omega^2 \end{aligned}\right\} \quad \bmod \theta^1, \dots, \theta^s \; . \tag{3.28}$$

The Pfaffian relations $\theta^1 = \cdots = \theta^{s-1} = 0$ then define the distribution $\mathcal{D}_1$. Now, we are assuming that $\mathcal{D}_1$ is nowhere-integrable, i.e., that $[\mathcal{D}_1, \mathcal{D}_1]$ properly contains $\mathcal{D}_1$ at every point. It follows that we have structure equations

$$d\theta^\alpha \equiv \theta^s \wedge (B_1^\alpha\,\omega^1 + B_2^\alpha\,\omega^2) \mod \theta^1, \ldots, \theta^{s-1} \quad \text{for all } 1 \le \alpha \le s-1. \tag{3.29}$$

where not all of the functions B_i^α vanish simultaneously. Of course, in the generic situation, the $(s-1)$-by-2 matrix $B = (B_i^\alpha)$ will have rank 2, but, for our purposes, we only need to assume that it has rank at least 1.

The upshot of all of these calculations is that we have *normalized structure equations* of the following form (where the greek index now runs over the range $1 \le \alpha, \beta \le s-1$):

$$\begin{aligned} d\theta^\alpha &= -\phi_\beta^\alpha \wedge \theta^\beta + \theta^s \wedge (B_1^\alpha\,\omega^1 + B_2^\alpha\,\omega^2) \\ d\theta^s &= -\phi_\beta^s \wedge \theta^\beta - \phi_s^s \wedge \theta^s + \omega^1 \wedge \omega^2, \end{aligned} \tag{3.30}$$

We will now use Eqs. (3.30) to describe the non-regular $\mathcal{D}$-curves which lie in U. Since all the bundles involved are trivial over U, we have $Q_U^* = U \times \mathbb{R}^s$. Using coordinates $p_1, \ldots, p_s$ on the $\mathbb{R}^s$-factor, we can write the canonical 2-form Ψ in the form

$$\begin{aligned} \Psi = d\big(p_1\,\theta^1 + \cdots + p_s\,\theta^s\big) &= \big(dp_\alpha - p_\beta\,\phi_\alpha^\beta - p_s\,\phi_\alpha^s\big) \wedge \theta^\alpha \\ &\quad + \big(dp_s - p_s\,\phi_s^s - p_\beta B_i^\beta\,\omega^i\big) \wedge \theta^s + p_s\,\omega^1 \wedge \omega^2 \end{aligned} \tag{3.31}$$

It is clear from this equation that Ψ has maximum rank (and hence no characteristics) except along the hypersurface $p_s = 0$. On this hypersurface, but away from the (proper) sublocus where $p_\beta B_1^\beta = p_\beta B_2^\beta = 0$, the form Ψ clearly has characteristics defined by the following $2s$ Pfaffian equations:

$$dp_\alpha - p_\beta\,\phi_\alpha^\beta = \theta^\alpha = \theta^s = p_\beta B_i^\beta\,\omega^i = 0\ . \tag{3.32}$$

We will denote this Pfaffian system by $\mathcal{C}$.

Let $Q_1^* \subset Q^* \subset T^*M$ denote the subbundle which is the annihilator of $\mathcal{D}_1 \subset TM$. In the above trivialization of Q_U^* over U, this subbundle is simply the locus defined by $p_s = 0$. We will denote by $Q_1^\bullet \subset Q_1^*$ the (dense) open subset of Q_1^* which, in each local trivialization as above, is the complement of the locus $p_\beta B_1^\beta = p_\beta B_2^\beta = 0$. Although we used a local coframing to compute the Pfaffian system $\mathcal{C}$ of rank $2s$ on the $(2s+1)$-manifold $Q_1^\bullet$, the result is clearly independent of that choice of coframing and hence is well-defined globally on $Q_1^\bullet$.

We can now state our main theorem.

Theorem 3.1 *Let $\mathcal{D}$ be a non-integrable rank 2 distribution on a manifold M^{2+s}. Suppose further that the distribution $\mathcal{D}_1 = \big[\mathcal{D}, \mathcal{D}\big]$ (which has rank 3) is nowhere-integrable. Then the projection to M of any Ψ-characteristic in $Q_1^\bullet$ is a locally rigid $\mathcal{D}$-curve.*

Proof . The proof rests on the following lemma, which generalizes Engel normal form (valid for the case $s = 2$) to a normal form for distributions $\mathcal{D}$ satisfying the hypotheses of the theorem.

Lemma 3.1 *For each $\xi \in Q_1^\bullet$, let C_ξ be the characteristic curve of Ψ passing through ξ. Let $\bar{C}_\xi$ be the image of C_ξ under the natural submersion $Q_1^\bullet \to M$. Then there is an* *open neighborhood U of the basepoint m of ξ on which there exists a local coordinate chart $(w, x, y, z, v_1, \dots, v_{s-2})$ centered on m with the following two properties. First, the component of $\bar{C}_\xi \cap U$ which contains m is described by*

$$x = y = z = v_1 = \cdots = v_{s-2} = 0.$$

Second, there are functions $F_1, \dots, F_{s-2}$ on U so that $\mathcal{D}$ is spanned in U by the vector fields

$$W = \frac{\partial}{\partial w} \qquad \text{and} \qquad X = \frac{\partial}{\partial x} + z\,\frac{\partial}{\partial y} + w\,\frac{\partial}{\partial z} + F_1\,\frac{\partial}{\partial v_1} + \cdots + F_{s-2}\,\frac{\partial}{\partial v_{s-2}}.$$

Assuming Lemma 3.1 for the moment, we will now prove that $\bar{C}_\xi$ is locally rigid in a neighborhood of m. Indeed, consider any compact segment $S \subset \bar{C}_\xi$ defined, in the local chart of the Lemma, by the set of relations $a \le w \le b$ and $x = y = z = v_1 = \cdots = v_{s-2} = 0$. This segment can obviously be parametrized as $\gamma_0\colon [a, b] \to U \subset M$ in such a way that, in the normal coordinates of the Lemma, we have

$$\gamma_0(t) = (t, 0, \dots, 0)\ . \tag{3.33}$$

Clearly, γ_0 has a C^1-neighborhood $\mathcal{U}$ in $\Omega_{\mathcal{D}}\big(\gamma_0(a), \gamma_0(b)\big)$ so that every other $\mathcal{D}$-curve γ in $\mathcal{U}$ can be parametrized in the form

$$\gamma(t) = \big(t, x(t), y(t), z(t), v_1(t), \dots, v_{s-2}(t)\big)\ . \tag{3.34}$$

Since $\gamma \in \Omega_{\mathcal{D}}\big(\gamma_0(a), \gamma_0(b)\big)$, the functions $x(t), y(t), z(t), v_1(t), \dots, v_{s-2}(t)$ must vanish at the endpoints of the interval $a \le t \le b$ and also must satisfy the equations

$$\begin{aligned} y'(t) &= z(t)\, x'(t) \\ z'(t) &= t\, x'(t) \\ v_1'(t) &= F_1\big(t, x(t), \dots, v_{s-2}(t)\big)\, x'(t) \\ &\ \vdots \\ v_{s-2}'(t) &= F_{s-2}\big(t, x(t), \dots, v_{s-2}(t)\big)\, x'(t) \end{aligned} \tag{3.35}$$

However, by the proof of Prop. 3.2, which carries over verbatim to this case, the endpoint conditions together with the first two of these equations imply that $x(t)$, $y(t)$, and $z(t)$ vanish identically. Of course, the remaining equations in turn now imply that the $v_i(t)$ are all constants. Since they must also vanish at the endpoints, they must in fact be zero. Thus, the curve γ_0 is rigid, as desired. □

It remains to prove Lemma 3.1. It is not clear why one might guess this Lemma to be true, but the essential hint is to be found in Cartan (1915), which contains a study of the case $s = 3$ (the first interesting case). Our proof was inspired by this analysis.

Proof . (of Lemma 3.1) Let $\xi \in Q_1^\bullet$ be fixed. Let $m \in M$ be the basepoint of ξ, so that $\xi \in T_m^* M$. First, we fix an m-neighborhood U on which there is a coframing $\omega^1, \omega^2, \theta^1, \dots, \theta^s$ satisfying the normalized structure equations defined above. Since, as a vector bundle, the sections of $\big(Q_1^*\big)_U$ are spanned by the 1-forms $\theta^1, \dots, \theta^{s-1}$, we may use a change of normalized coframing to arrange that the point ξ correspond to the point $\big(m, (1, 0, \dots, 0)\big)$ under the identification $\big(Q_1^*\big)_U = U \times \mathbb{R}^{s-1}$ described above. In fact, under this identification, we have

$$\begin{aligned}\Psi &= d\big(p_1\,\theta^1 + \cdots + p_{s-1}\,\theta^{s-1}\big)\\ &= \big(dp_\alpha - p_\beta\,\phi^\beta_\alpha\big)\wedge\theta^\alpha + \big(-p_\beta\,B^\beta_i\,\omega^i\big)\wedge\theta^s,\end{aligned}\tag{3.36}$$

where, since $\xi = \big(m,(1,0,\dots,0)\big)$ is an element of $Q_1^\bullet$, the 1-form $B^1_1\,\omega^1 + B^1_2\,\omega^2$ must be non-zero at m. By shrinking our m-neighborhood U, we may assume that this 1-form is non-vanishing on U. It follows that the closed 2-form Ψ has Engel half-rank s on $\big(Q_1^*\big)_U$, which is a manifold of dimension $2s$+1.

Now, by Darboux' theorem, there must exist a neighborhood V of ξ in $\big(Q_1^*\big)_U$ and a submersion $F\colon V \to \mathbb{R}^{2s}$ so that $\Psi = F^*(\Omega_0)$, where Ω_0 is the standard symplectic structure on $\mathbb{R}^{2s}$. Note that the fibers of F are the characteristic curves of Ψ. Now, consider the subspace

$$E = \{\, v \in T_\xi\big(Q_1^*\big)_U \mid \big(dp_\alpha - p_\beta\,\phi^\beta_\alpha\big)(v) = 0,\ 1 \le \alpha \le s{-}1 \,\} \subset T_\xi\big(Q_1^*\big)_U\,.$$

This vector space has dimension s+2. Moreover, E has the property that Ψ, when restricted to E, becomes decomposable (i.e., of half-rank 1) as well as the property that it contains the Ψ-characteristic direction through ξ.

From this latter property, it follows that the image subspace $\overline{E} = F'(E) \subset T_{F(\xi)}\mathbb{R}^{2s}$ is of dimension s+1 while, from the former property, it follows that Ω_0 restricted to $\overline{E}$ has half-rank 1. Now, given these properties of $\overline{E}$, it is a standard fact of symplectic geometry (which follows from the Darboux-Weinstein Theorem) that $\overline{E}$ is the tangent space of a smooth submanifold $\overline{S} \subset \mathbb{R}^{2s}$ of dimension s+1 with the property that Ω_0 pulls back to $\overline{S}$ to be a closed decomposable 2-form.

Thus, if we let $S = F^{-1}\big(\overline{S}\big)$, then S is a smooth manifold of dimension s+2 with following properties: First, ξ lies on S and $T_\xi S = E$; second, the pull-back of Ψ to S is a closed non-zero decomposable 2-form; and third, S is foliated by characteristic curves of Ψ. In particular, by shrinking V, we may suppose that the component of $C_\xi \cap V$ which contains ξ actually lies in S.

Since E is transverse to the fibers of the submersion $Q_1^\bullet \to M$, it follows that, by shrinking U again if necessary, we may suppose that S is the image of a smooth section θ of Q_1^* over U and hence is described by equations of the form $p_\alpha = f_\alpha$ where the f_α are functions on U. Since ξ lies on S, these functions must satisfy $f_1(m) = 1$ and $f_\alpha(m) = 0$ for $1 < \alpha < s$. By making a change of coframing, replacing θ^1 by $f_1\,\theta^1 + \cdots + f_{s-1}\,\theta^{s-1}$, we may clearly arrange it so that S is described by the equations $p_1 = 1$ and $p_\alpha = 0$, so we do this.

Now, by construction, the 2-form

$$d\theta^1 = -\phi^1_\alpha \wedge \theta^\alpha - \big(B^1_1\omega^1 + B^1_2\omega^2\big)\wedge\theta^s \tag{3.37}$$

is decomposable. Also, by construction, we have $d\theta^1 \equiv 0 \bmod \theta^1,\dots,\theta^s$ while $d\theta^1 \not\equiv 0 \bmod \theta^1,\dots,\theta^{s-1}$. It follows that $d\theta^1$ has a linear factor which lies in the span of $\theta^1,\dots,\theta^s$ but not in the span of $\theta^1,\dots,\theta^{s-1}$. Thus, $d\theta^1$ must have a factor of the form $\theta^s + g_1\,\theta^1 + \cdots + g_{s-1}\,\theta^{s-1}$. Again, by changing coframe, replacing θ^s by $\theta^s + g_1\,\theta^1 + \cdots + g_{s-1}\,\theta^{s-1}$, we may arrange that θ^s itself is a factor of $d\theta^1$. Finally, by making a suitable coframe change involving ω^1 and ω^2, we can arrange to have

$$d\theta^1 = -\theta^s \wedge \omega^1\,. \tag{3.38}$$

By the Pfaff-Darboux theorem (Bryant, et al, 1991, Thm. II.3.4), it follows that there exist functions x, y, and z on a neighborhood of m so that $\theta^1 = dy - z\,dx$. It is easy to see that, by taking sufficient care, we may arrange that $\theta^s \wedge dx \neq 0$ and that x, y, and z vanish at m. Since

$$d\theta^1 = -dz \wedge dx = -\theta^s \wedge \omega^1 \ , \tag{3.39}$$

it follows that $\theta^s = \lambda\big(dz - w\,dx\big)$ for some non-zero function λ and some function w on U. Replacing θ^s, ω^1, and ω^2 respectively by $(1/\lambda)\theta^s$, $\lambda\,\omega^1$, and $(1/\lambda^2)\omega^2$, we may keep all of our structure equations so far and arrange that $\theta^s = dz - w\,dx$. Replacing y, z, and w, respectively, by $y - \frac{1}{2}w(m)x^2$, $z - w(m)x$, and $w - w(m)$, we may even arrange that the function w vanishes at m. Finally, by adding a suitable multiple of θ^s to ω^1, we may arrange that $\omega^1 = dx$.

The structure equations

$$\begin{aligned} d\theta^1 &= -dz \wedge dx = -\theta^s \wedge \omega^1 \\ d\theta^s &= dx \wedge dw \equiv \omega^1 \wedge \omega^2 \bmod \theta^1, \ldots, \theta^s \end{aligned} \tag{3.40}$$

now show that the functions w, x, y, and z all have linearly independent differentials on U and that we have

$$dw \equiv \omega^2 \bmod \theta^1, \theta^s, dx \ . \tag{3.41}$$

Clearly, we may modify ω^2 so as to have $\omega^2 = dw$ without affecting any of our previous normalizations. Thus, from now on, we assume that $\omega^2 = dw$.

We must still find $s-2$ more functions to complete our desired coordinate system. To do this, let us return to the consideration of the foliation of S by characteristic curves of Ψ. Since θ^1, regarded as a section of Q_1^* has its image equal to S, we can use $\theta^1 \colon U \to S$ to pull back this characteristic foliation to a foliation $\mathcal{F}$ of U by $\mathcal{D}$-curves. By Prop. 2.1, this foliation consists of non-regular $\mathcal{D}$-curves. In fact, by our constuction, the leaf of $\mathcal{F}$ passing through m is precisely the component of $\overline{C}_\xi \cap U$ containing m. Since these leaves are $\mathcal{D}$-curves, all of the forms θ^α vanish on the leaves of $\mathcal{F}$. Also by our construction, the leaves of $\mathcal{F}$ are characteristic for θ^1, and, since $\theta^1 \wedge d\theta^1 = -dx \wedge dy \wedge dz$, it follows that the functions x, y, and z are constant on the leaves of $\mathcal{F}$. In particular, $\omega^1 = dx$ vanishes on the leaves of $\mathcal{F}$. It follows that $\omega^2 = dw$ must pull back to each of the leaves of $\mathcal{F}$ to be non-zero since this is the only 1-form left in the coframing which does not vanish on the leaves of $\mathcal{F}$.

Now, shrinking U again if necessary, we may choose, beyond x, y, and z, an additional $s-2$ independent functions $v_1, \ldots, v_{s-2}$ vanishing at m which are also constant on the leaves of $\mathcal{F}$. Since dw does not vanish on the leaves of $\mathcal{F}$, it follows that the differentials of the $s+2$ functions $w, x, y, z, v_1, \ldots, v_{s-2}$ are independent. Hence they form a local coordinate system centered on m.

Finally, since the foliation $\mathcal{F}$ is defined by the Pfaffian system

$$dx = dy = dz = dv_1 = \cdots = dv_{s-2} = 0 \ , \tag{3.42}$$

and consists of $\mathcal{D}$-curves, and since we know that dx is linearly independent from the forms $\theta^1, \ldots, \theta^s$, it easily follows that there must exist functions $F_1, \ldots, F_{s-2}$ on U so that, on U the system $\mathcal{D}$ is defined by the Pfaffian equations

$$dy - z\,dx = dz - w\,dx = dv_1 - F_1\,dx = \cdots = dv_{s-2} - F_{s-2}\,dx = 0 \ . \tag{3.43}$$

However, this is exactly the statement of the lemma. $\square$

The referee asked if it wasn't true that one could arrange that all of the functions F_i vanish at m. In fact, one can easily arrange this to be true. If one notices that

$$d\big(v_i - F_i(m)\,x\big) - \big(F_i - F_i(m)\big)\,dx = dv_i - F_i\,dx$$

and simply uses the functions $\tilde{v}_i = v_i - F_i(m)\,x$ instead of the original functions v_i, this will not destroy any of the desired properties of the original coordinate system and will replace each function F_i by $F_i - F_i(m)$ in the normal form. It is conceivable that this vanishing condition may be desirable in some situations.

It is interesting to note that the local normal form provided for by Lemma 3.1 depends on a choice of $s{-}2$ functions of $s{+}2$ variables, namely the functions $F_1, \ldots, F_{s-2}$. Here is why this should be expected: A rank 2 distribution on a manifold of dimension $s{+}2$ is determined by a section of the Grassmannian bundle $G_2(TM) \to M$ of 2-planes in tangent spaces to M. The fibers of this bundle are Grassmannian manifolds of the form $G_2\big(\mathbb{R}^{s+2}\big)$ and hence have dimension $2s$. Thus, the local rank 2 distributions depend on $2s$ functions of $s{+}2$ variables. Since the local diffeomorphisms in dimension $s{+}2$ depend on $s{+}2$ functions of $s{+}2$ variables, it should be expected that the local diffeomorphism class of a rank 2 distribution in $\mathbb{R}^{s+2}$ should depend on $2s - (s{+}2) = s{-}2$ functions of $s{+}2$ variables. Of course, this is exactly what our normal form produces. Thus, this normal form is, in some sense, optimal.

Our main theorem does not say anything about the non-regular curves which have a lifting to a Ψ-characteristic which happens to lie in $Q_1^* \setminus Q_1^\bullet$. It is an interesting and open problem to determine whether, under the bracket generating assumption, *all* of the non-regular curves are rigid.

4. Examples

In this section, we are going to consider some examples which illustrate the general theory of the previous sections. First, let us introduce some notation. For any distribution $\mathcal{D}$ on a manifold M, we define $\mathcal{D}_1 = [\mathcal{D}, \mathcal{D}]$ and, by induction, $\mathcal{D}_{i+1} = [\mathcal{D}_i, \mathcal{D}_i]$, for all $i \geq 1$.

4.1 Goursat systems

E. Goursat [1923] made a study of rank 2 distributions $\mathcal{D}$ on M^{2+s} with the property that each of the distributions $\mathcal{D}_i$ for $0 \leq i \leq s$ was of constant rank $2{+}i$. Note that, for $s > 2$ this is not "generic" behavior. For example, when $s > 2$, a "generic" rank 2 distribution $\mathcal{D}$ will have distributions $\mathcal{D}_1$ and $\mathcal{D}_2$ be of ranks 3 and 5, respectively.[6]

In any case, for distributions $\mathcal{D}$ satisfying this hypothesis, Goursat derived the following result (which he attributed to von Weber): There exists a closed "exceptional" set $E \subset M$ with no interior, so that $M \setminus E$ can be covered by open sets U on which there exist coordinates $(x, y_0, y_1, \ldots, y_s)$ so that, in U, the distribution $\mathcal{D}$ is described by the Pfaffian equations

$$dy_0 - y_1\,dx = dy_1 - y_2\,dx = \cdots = dy_{s-1} - y_s\,dx = 0\,. \tag{4.1}$$

Note that, except for the caveat about the exceptional set, this is a generalization of Engel's normal form.[7]

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[6] The referee has pointed out to us that, in the control theory literature, a Goursat system is sometimes referred to as an "integrator".

[7] For historical accuracy, it should be noted that Goursat did not notice that his proof of the stated normal form failed on an exceptional set. This was first noticed by Giaro, et al (1978), who provided an example

In an open set U in which the system can be placed in this normal form, it is easy to describe the rigid curves which arise as the projections of Ψ-characteristics in $Q_1^\bullet$. They are simply the curves of the foliation described by

$$dx = dy_0 = dy_1 = \cdots = dy_{s-1} = 0 \ . \tag{4.2}$$

Moreover, any $\mathcal{D}$-curve on which dx is not identically zero is easily seen to be regular. Thus, the non-regular $\mathcal{D}$-curves of the Goursat system (1) form a foliation of $\mathbb{R}^{2+s}$.

This suggests that the following proposition might be true.

Proposition 4.1 *Let $\mathcal{D}$ be a rank 2 distribution on M^{2+s} and suppose that $\mathcal{D}_1$ and $\mathcal{D}_2 = [\mathcal{D}_1, \mathcal{D}_1]$ have constant ranks, respectively, of 3 and 4. Then the Ψ-characteristic curves in $Q_1^\bullet$ project to M to define a foliation $\mathcal{F}$ of M by locally rigid $\mathcal{D}$-curves.*

Proof . The hypotheses on $\mathcal{D}$ make it possible to choose, for any point $m \in M$, an m-neighborhood U and a coframing $\omega^1, \omega^2, \theta^1, \ldots, \theta^s$ on U, so that, on U, we have, first, that $\mathcal{D}$ is defined by the Pfaffian equations $\theta^1 = \cdots = \theta^s = 0$; second, that $\mathcal{D}_1$ is defined by the Pfaffian equations $\theta^1 = \cdots = \theta^{s-1} = 0$; and, third, that $\mathcal{D}_2$ is defined by the Pfaffian equations $\theta^1 = \cdots = \theta^{s-2} = 0$. Moreover, we have the congruences

$$\begin{aligned} d\theta^\alpha &\equiv \quad 0 \quad \bmod \theta^1, \ldots, \theta^{s-1}, && 1 \le \alpha \le s-2, \\ d\theta^\alpha &\equiv \quad 0 \quad \bmod \theta^1, \ldots, \theta^{s} \quad , && 1 \le \alpha \le s-1, \\ d\theta^s &\equiv A\,\omega^1 \wedge \omega^2 \bmod \theta^1, \ldots, \theta^{s} \quad , && \end{aligned} \tag{4.3}$$

where A is a non-zero function. By replacing ω^1 by an appropriate multiple, we can assume that $A = 1$. Moreover, since $\mathcal{D}_1$ is not completely integrable, it follows that we must have

$$d\theta^{s-1} \equiv (a_1\,\omega^1 + a_2\,\omega^2) \wedge \theta^s \bmod \theta^1, \ldots, \theta^{s-1} \ , \tag{4.4}$$

where a_1 and a_2 are not both zero. By making a unimodular basis change among ω^1 and ω^2 (so as not to disturb the $d\theta^s$ congruence), we may assume that $a_1 = 1$ and $a_2 = 0$. Thus, our structure equations take the form

$$\begin{aligned} d\theta^\alpha &\equiv -\phi^\alpha_\beta \wedge \theta^\beta - \phi^\alpha_{s-1} \wedge \theta^{s-1} \\ d\theta^{s-1} &\equiv -\phi^{s-1}_\beta \wedge \theta^\beta - \phi^{s-1}_{s-1} \wedge \theta^{s-1} + \omega^1 \wedge \theta^s \\ d\theta^s &\equiv -\phi^s_\beta \wedge \theta^\beta - \phi^s_{s-1} \wedge \theta^{s-1} - \phi^s_s \wedge \theta^s + \omega^1 \wedge \omega^2 \end{aligned} \tag{4.5}$$

where we are now restricting the greek index to the range $1 \le \alpha \le s-2$. We shall say that a coframing $\theta^1, \ldots, \theta^s, \omega^1, \omega^2$ on a domain $U \subset M$ is *adapted* to $\mathcal{D}$ if $\theta^1 = \cdots = \theta^s = 0$ defines the distribution $\mathcal{D}$ in U and, moreover, Eqs. (4.5) hold.

It is easy to see that the Pfaffian system spanned by $\theta^1, \ldots, \theta^s, \omega^1$ is independent of the choice of $\mathcal{D}$-adapted coframing in an open set U. Of course, it follows that there is a Pfaffian system $\mathcal{J}$ globally well-defined on M which, relative to a $\mathcal{D}$-adapted coframing in U, is spanned by the 1-forms $\theta^1, \ldots, \theta^s, \omega^1$. We will let $\mathcal{F}$ denote the foliation of M by the integral curves of $\mathcal{J}$. Of course, relative to a $\mathcal{D}$-adapted coframing in U these are defined by the Pfaffian equations

with $s = 3$ where this exceptional set was non-empty. Their example was the system on $\mathbb{R}^5$ defined by the equations $dy_0 - y_1\,dx = dy_1 - y_2\,dx = z\,dy_2 - dx = 0$. In this case, the exceptional set is the hyperplane $z = 0$; away from this locus, setting $y_3 = 1/z$ puts the system in Goursat normal form. As we shall see, this example actually fits quite nicely into the context of Cartan's result, Thm. 4.1.

$$\theta^1 = \cdots = \theta^s = \omega^1 = 0\ . \tag{4.6}$$

Let us now fix a $\mathcal{D}$-adapted coframing in an open set U. Then, in the standard identification $(Q_1^*)_U = U \times R^{s-1}$ discussed in §2, the subset $(Q_1^\bullet)_U$ is described by the inequality $p_{s-1} \neq 0$. Our formula for Ψ becomes

$$\begin{aligned}\Psi &= d\big(p_\alpha\,\theta^\alpha + p_{s-1}\,\theta^{s-1}\big)\\ &= \big(dp_\alpha - p_\beta\,\phi^\beta_\alpha - p_{s-1}\,\phi^{s-1}_\alpha\big)\wedge\theta^\alpha\\ &\quad + \big(dp_{s-1} - p_\beta\,\phi^\beta_{s-1} - p_{s-1}\,\phi^{s-1}_{s-1}\big)\wedge\theta^{s-1} + p_{s-1}\,\omega^1\wedge\theta^s\end{aligned} \tag{4.7}$$

It follows from this that the Ψ-characteristics on $\big(Q_1^\bullet\big)_U$ are defined by the Pfaffian equations

$$\begin{aligned}\theta^\alpha = \theta^{s-1} = \theta^s = \omega^1 = 0\ ,\\ \big(dp_\alpha - p_\beta\,\phi^\beta_\alpha - p_{s-1}\,\phi^{s-1}_\alpha\big) = 0\ ,\\ \big(dp_{s-1} - p_\beta\,\phi^\beta_{s-1} - p_{s-1}\,\phi^{s-1}_{s-1}\big) = 0\ .\end{aligned} \tag{4.8}$$

Equation (4.8) makes it clear that the Ψ-characteristic curves project to be the leaves of $\mathcal{F}$ and that, conversely, every leaf of $\mathcal{F}$ is the projection of a ($(s-1)$-parameter family of) Ψ-characteristic curve(s).

4.2 Prolongation and Deprolongation

At this point, it is worth discussing a result of Cartan (1914), which gives a local structure theorem for rank 2 systems satisfying the conditions of Prop. 4.1. First, we recall the definition of *prolongation* in this context.

If $\mathcal{D}$ is a rank 2 distribution on a manifold M^{2+s}, then, regarding $\mathcal{D}$ as a vector bundle, we can certainly define its projectivization $\pi\colon \mathbb{P}\mathcal{D} \to M$, which is a bundle over M whose typical fiber $\mathbb{P}\mathcal{D}_m$ is the space of 1-dimensional linear subspaces of the 2-dimensional vector space $\mathcal{D}_m$. Thus, the fibers of $\mathbb{P}\mathcal{D}$ are isomorphic to $\mathbb{RP}^1$ as projective 1-manifolds. There is a canonical rank 2 distribution $\mathcal{D}^{(1)}$ on $\mathbb{P}\mathcal{D}$ defined by setting $\mathcal{D}^{(1)}_\xi = (\pi')^{-1}\big(\xi\big)$ for each linear subspace $\xi \subset \mathcal{D}_m$. The distribution $\mathcal{D}^{(1)}$ is called the *(first) prolongation* of $\mathcal{D}$.

For any immersed $\mathcal{D}$-curve $\gamma\colon [a,b] \to M$, there is a canonical lift $\gamma^{(1)}\colon [a,b] \to \mathbb{P}\mathcal{D}$ defined by letting $\gamma^{(1)}(t) = \mathbb{R}\cdot\gamma'(t)$. It is easy to verify from the very definition that $\gamma^{(1)}$ is an immersed $\mathcal{D}^{(1)}$-curve which is transverse to the fibers of π and that, conversely, every immersed $\mathcal{D}^{(1)}$-curve $\psi\colon [a,b] \to \mathbb{P}\mathcal{D}$ which is transverse to the fibers of π is of the form $\psi = \gamma^{(1)}$ for a unique immersed $\mathcal{D}$-curve γ.

Thus, with the obvious exception of the fibers of π, which are clearly $\mathcal{D}$-curves, "almost all" $\mathcal{D}^{(1)}$-curves are in one-to-one correspondence with "almost all" $\mathcal{D}$-curves.

Now, it is easy to show that $\mathcal{D}^{(1)}_1 = \big[\mathcal{D}^{(1)},\mathcal{D}^{(1)}\big] = \pi^*\big(\mathcal{D}\big)$ and that $\mathcal{D}^{(1)}_2 = \big[\mathcal{D}^{(1)}_1,\mathcal{D}^{(1)}_1\big] = \pi^*\big(\mathcal{D}_1\big)$. It follows that, if $\mathcal{D}_1$ has rank 3 everywhere, then $\mathcal{D}^{(1)}_1$ has rank 3 and $\mathcal{D}^{(1)}_2$ has rank 4. Thus, the prolongation of a generic rank 2 system satisfies the conditions of Prop. 4.1. It should come as no surprise (and, in any case, is easy to check) that the canonical foliation $\mathcal{F}$ constructed in Prop. 4.1 is, in the case of $\mathcal{D}^{(1)}$, merely the foliation by the fibers of π.

The following result shows that, in fact, every distribution satisfying the hypotheses of Prop. 4.1 is locally of the form $\mathcal{D}^{(1)}$ for some distribution $\mathcal{D}$.

Theorem 4.1 (Cartan, 1914) *Let $\mathcal{D}$ be a rank 2 distribution on a manifold M^{s+2} and suppose that $\mathcal{D}_1$ and $\mathcal{D}_2$ have ranks 3 and 4 respectively. Furthermore, suppose that there is a submersion $f\colon M \to N^{s+1}$ with the property that the fibers of f are the leaves of the canonical foliation $\mathcal{F}$. Then there exists a unique rank 2 distribution $\mathcal{D}'$ on N with the property that $\mathcal{D}_1 = f^*(\mathcal{D}')$ and, moreover, there exists a canonical smooth map $f^{(1)}\colon M \to \mathbb{P}\mathcal{D}'$ which is a local diffeomorphism, which satisfies $f = \pi \circ f^{(1)}$, and which satisfies $\mathcal{D} = \left(f^{(1)}\right)^{-1}\left((\mathcal{D}')^{(1)}\right)$.*

We will not give the proof of this result here, instead referring the reader either to Cartan or to the more modern exposition in Sluis (1992). Alternatively, one can simply verify that the following definitions work: The distribution $\mathcal{D}'$ is defined by the rule

$$\mathcal{D}'_{f(x)} = f'(x)\big((\mathcal{D}_1)_x\big) \tag{4.9}$$

and the map $f^{(1)}$ is defined by $f^{(1)}(x) = f'(x)\big(\mathcal{D}_x\big)$. The only slightly subtle point is that one must use that the fibers of f are connected in order to show that this well-defines $\mathcal{D}'$. Note that every point of M lies in a neighborhood U which does have a submersion whose fibers are the leaves of $\mathcal{F}$ restricted to U, so this theorem can always be applied locally. Indeed it is the basis of the proof of Goursat's theorem.

If $\mathcal{D}$ is a rank 2 distribution with the property that the systems $\mathcal{D}_i$ (as defined earlier in this section) all have constant rank, then there is a largest integer ℓ for which the rank of $\mathcal{D}_\ell$ is ℓ+2. According to Gardner (1967), the rank of $\mathcal{D}_{\ell+1}$ is then either ℓ+2 (in which case $\mathcal{D}_{\ell+1} = \mathcal{D}_\ell$ and the distribution $\mathcal{D}_\ell$ is a Frobenius system) or else is ℓ+4 (which is, in some sense, the "generic" situation). In this latter case, it follows that $\mathcal{D}$ can be locally "deprolonged" ($\ell-1$) times to a rank 2 distribution $\overline{\mathcal{D}}$ on a manifold $N^{2+s-(\ell-1)}$ and this $\overline{\mathcal{D}}$ is not the prolongation of a rank 2 distribution on a lower dimensional manifold.

It is interesting to note that, when $\ell \geq 2$, there is a distinguished subset $E \subset M$ corresponding to the points $x \in M$ such that $f^{(1)}(x)$ is tangent to the canonical foliation $\mathcal{F}'$ associated to the distribution $\mathcal{D}'$ on N. When $s = 5$, this set is precisely the "exceptional set" on which Goursat's normal form fails. A similar description of the exceptional set can be given for larger s as well.

The end result of this is that in the constant rank case one either has a Goursat system or else by a process of "de-prolonging" one can always reduce to the case where the rank of $\mathcal{D}_1$ is 3 and the rank of $\mathcal{D}_2$ is 5. In this latter case, we have the following analog of Prop. 4.1.

Proposition 4.2 *Let $\mathcal{D}$ be a rank 2 distribution on M^{2+s} and suppose that $\mathcal{D}_1$ and $\mathcal{D}_2 = [\mathcal{D}_1, \mathcal{D}_1]$ have constant ranks, respectively of 3 and 5. Then, for each 1-dimensional subspace $\xi \subset \mathcal{D}_m$, there is an $(s-1)$-parameter family of non-homothetic Ψ-characteristic curves in $Q_1^\bullet$ which project to M so that they pass through m tangent to ξ.*

4.3 Systems of Cartan type

Cartan (1910) contains a thorough study of rank 2 distributions $\mathcal{D}$ on 5-manifolds which have the property that $\mathcal{D}_1$ has rank 3 and $\mathcal{D}_2$ has (the maximum) rank 5. It is easy to see that these rank assumptions hold almost everywhere for a generic rank 2 distribution on a 5-manifold. We shall call these distributions *systems of Cartan type*.

For any system of Cartan type, the bundle $Q_1^\bullet$ is simply Q_1 minus its zero section. As a result *all* of the non-regular immersed $\mathcal{D}$-curves are projections of Ψ-characteristic curves in $Q_1^\bullet$. It is then a consequence of Prop. 4.2 and a dimension count that there is exactly a 5-parameter family of non-regular $\mathcal{D}$-curves and that they are all locally rigid. Moreover, there is a unique non-regular curve through each point in each direction tangent to the distribution $\mathcal{D}$.

It follows that there is a sort of "projective exponential" surface Σ_p associated to each point $p \in M^5$ which is generated by the 1-parameter family of non-regular curves through p. The geometry of how these surfaces sit in M is quite interesting. For example, by analysing them, it can be shown that, if M is connected, then any two points of M can be joined by a piecewise smooth $\mathcal{D}$-curve whose smooth segments are rigid. Thus, the "piecewise rigid" curves are always present in any variational problem for $\mathcal{D}$-curves joining any two points of M.

In Cartan (1910) it is shown that one can associate a connection and consequently a curvature to any distribution $\mathcal{D}$ of Cartan type. The fundamental curvature tensor of this connection turns out to be a homogenous quartic form F on $\mathcal{D}$, i.e., a section of $S^4(\mathcal{D}^*)$.

Cartan showed that, when F vanishes identically, then every point of M has an open neighborhood U on which there exists a coordinate chart (w, x, y, z, v) in which $\mathcal{D}$ is described by the Pfaffian equations

$$dy - z\,dx = dz - w\,dx = dv - w^2\,dx = 0\ . \tag{4.10}$$

Moreover, he showed that the Lie algebra of infinitesimal symmetries of this $\mathcal{D}$ has the largest possible dimension of any system of Cartan type and was isomorphic to the 14-dimensional exceptional Lie algebra of non-compact type $\mathfrak{g}_2$. It is interesting to note that this very example was written down by Hilbert (1912) as an example of a system whose integral curves could not be expressed in terms of an aribtrary function and a finite number of derivatives. That no such formula existed was interesting because it showed that the problem of constructing local, fixed-endpoint variations of the corresponding $\mathcal{D}$-curves was non-trivial.

The quartic form F is probably the analog for systems of Cartan type of the Ricci tensor in Riemannian geometry. For example, it seems that when F is everywhere positive definite and M is "$\mathcal{D}$-complete" in an appropriate sense, then M must be compact. However, the analysis of this geometry is rather complicated and will be postponed to a later paper devoted to the geometry of rigid curves in systems of Cartan type.

4.4 Rolling Surfaces

We will content ourselves with studying one geometric case where systems of Cartan type arise, the case of the mechanical system represented by rolling one surface over another without slipping or twisting. Special cases of this, usually a sphere rolling over a plane or, more generally, over a surface in $\mathbb{E}^3$, have been mentioned in the literature (cf. Arnold (1989, p. 96) or the recent preprint by Brockett and Dai). Our treatment will be more general and abstract.

In usual formulations of the problem, one starts with a stationary surface Σ_1 and a moveable surface Σ_2, imagined embedded in Euclidean space. The state space is then described by choosing two points $p_1 \in \Sigma_1$ and $p_2 \in \Sigma_2$ and an isometric identification ι of the tangent spaces,

$$\iota: T_{p_1}\Sigma_1 \to T_{p_2}\Sigma_2 .$$

Geometrically, one should imagine that the surface Σ_1 is stationary and that one moves the surface Σ_2 into tangential contact with Σ_1 so that p_2 is coïncident with p_1 and then rotates and/or reflects Σ_2 so that the desired identification of the tangent spaces is achieved. It is clear that the space of triples (p_1, p_2, ι) forms a manifold M of dimension 5. We are going to describe the canonical rank 2 distribution $\mathcal{D}$ on M which has the property that $\mathcal{D}$-curves represent the possible ways of "rolling" Σ_2 over Σ_1 without slipping or twisting.

It is easy to see that a curve $\gamma: [a, b] \to M$ which represents such a motion must be of the form

$$\gamma(t) = \big(u_1(t), u_2(t), \iota(t)\big) \tag{4.11}$$

where $u_i: [a, b] \to \Sigma_i$ are smooth curves with the property that $\iota(t)\big(u_1'(t)\big) = u_2'(t)$ for all $a \le t \le b$. This captures the property of "rolling without slipping". However, this is not sufficient to prevent "twisting". The "twisting" condition is easily encoded as follows: If we let $e_1, f_1: [a, b] \to T\Sigma_1$ denote a parallel orthonormal frame field along the curve u_1, then we require that the corresponding frame field $e_2, f_2: [a, b] \to T\Sigma_2$ defined by

$$e_2(t) = \iota(t)\big(e_1(t)\big) \qquad\qquad f_2(t) = \iota(t)\big(f_1(t)\big) \tag{4.12}$$

should also be parallel (along u_2).

It is clear from this formulation that no reference need be made to the actual embeddings of the surfaces in Euclidean space. In other words, this problem could just as well be considered for an arbitrary pair of abstract surfaces endowed with Riemannian metrics, so this is what we shall do. For simplicity of the exposition, we are going to restrict to the case of orientable surfaces and oriented identifications, but this is a trivial restriction.

Thus, let Σ_1 and Σ_2 be oriented surfaces endowed with Riemannian metrics $d\sigma_1^2$ and $d\sigma_2^2$ respectively. Let F_1 (respectively, F_2) denote the oriented orthonormal frame bundle of Σ_1 (respectively, Σ_2) with respect to its induced metric.

As usual (see any elementary book on the Riemannian geometry of surfaces), there are canonical 1-forms α_1, α_2, and α_{21} ($= -\alpha_{12}$) on F_1 and corresponding 1-forms β_1, β_2, and β_{21} ($= -\beta_{12}$) on F_2 satisfying the structure equations

$$\begin{aligned}
d\alpha_1 &= \alpha_{21} \wedge \alpha_2 & \qquad d\beta_1 &= \beta_{21} \wedge \beta_2 \\
d\alpha_2 &= -\alpha_{21} \wedge \alpha_1 & \qquad d\beta_2 &= -\beta_{21} \wedge \beta_1 \\
d\alpha_{21} &= A\,\alpha_1 \wedge \alpha_2 & \qquad d\beta_{21} &= B\,\beta_1 \wedge \beta_2
\end{aligned} \tag{4.13}$$

where A (respectively, B) is the Gauss curvature of the metric $d\sigma_1^2$ (respectively, $d\sigma_2^2$).

Now, each of F_1 and F_2 are principal right SO(2)-bundles. Let SO(2) act diagonally on $F_1 \times F_2$ (this is, of course, a free action) and set $M = (F_1 \times F_2)/\mathrm{SO}(2)$. A moment's thought shows that an element of the 5-manifold M has a natural interpretation as a triple (p_1, p_2, ι) where $\iota: T_{p_1}\Sigma_1 \to T_{p_2}\Sigma_2$ is an oriented isometry, so M is actually the "state space" of our desired mechanical system.

Let $\tilde{\mathcal{D}}$ be the rank 3 distribution on $F_1 \times F_2$ defined by the Pfaffian equations

$$\alpha_1 - \beta_1 = \alpha_2 - \beta_2 = \alpha_{21} - \beta_{21} = 0\,. \tag{4.14}$$

Note that $\tilde{\mathcal{D}}$ is invariant under the diagonal SO(2)-action on $F_1 \times F_2$ and that it contains the tangents to the fibers of the submersion $F_1 \times F_2 \to M$. It follows that there is a well-defined rank 2 distribution $\mathcal{D}$ on M which is the "push-down" of the distribution $\tilde{\mathcal{D}}$.

Now, it is clear that any $\tilde{\mathcal{D}}$-curve $\tilde{\gamma}\colon [a,b] \to F_1 \times F_2$ is of the form

$$\tilde{\gamma}(t) = \big((p_1(t); e_1(t), f_1(t)),\ (p_2(t); e_2(t), f_2(t)) \big) \tag{4.15}$$

where each (e_i, f_i) is an oriented orthonormal frame field along $p_i\colon [a,b] \to \Sigma_i$, where there exist functions g and h on the interval $[a,b]$ so that, for $i = 1$ or 2, we have $p_i'(t) = g(t)\, e_i(t) + h(t)\, f_i(t)$, and, moreover, where the "rotation rates" of the two frame fields along their respective base curves are the same at each time t. It follows that the quotient curve $\gamma\colon [a,b] \to M$ of such a $\tilde{\gamma}$ represents a "rolling-without-twisting-or-slipping" of one of the surfaces over the other. Conversely every such "rolling-without-twisting-or-slipping" clearly arises in this way. It follows that a motion of rolling without twisting or slipping corresponds exactly to a $\mathcal{D}$-curve. Thus, $\mathcal{D}$ describes the "non-holonomic" constraints of our mechanical system.

Let us set

$$\begin{aligned} \theta^1 &= \tfrac{1}{2}\big(\alpha_1 - \beta_1\big) \\ \theta^2 &= \tfrac{1}{2}\big(\alpha_2 - \beta_2\big) \\ \theta^3 &= \tfrac{1}{2}\big(\alpha_{21} - \beta_{21}\big) \end{aligned} \tag{4.16}$$

and also set $\omega^1 = \frac{1}{2}\big(\alpha_1 + \beta_1\big)$ and $\omega^2 = \frac{1}{2}\big(\alpha_2 + \beta_2\big)$. We easily compute that

$$\left.\begin{aligned} d\theta^1 &\equiv \ \ \theta^3 \wedge \omega^2 \\ d\theta^2 &\equiv -\theta^3 \wedge \omega^1 \end{aligned}\right\} \quad \operatorname{mod} \theta^1, \theta^2 \tag{4.17}$$

and, moreover, that

$$d\theta^3 \equiv \tfrac{1}{2}\big(B - A\big)\, \omega^1 \wedge \omega^2 \ \operatorname{mod} \theta^1, \theta^2, \theta^3\ . \tag{4.18}$$

It follows that, on the open set in M where $A - B \neq 0$, the distribution $\mathcal{D}$ is of Cartan type.

Straightforward computation using Prop. 2.1 now shows that, on the open set where $A - B \neq 0$, the 5-parameter family of rigid curves (not surprisingly) describes the motion of rolling Σ_2 along Σ_1 in such a way that each of the contact curves p_i traces out a geodesic in Σ_i. (Of course, if $(B - A)$ vanishes identically on M, then the Gaussian curvatures of the two surfaces must not only be equal, they must be constant. In this case, the distribution $\mathcal{D}$ is completely integrable. Its leaves correspond to the (local) isometries of the two surfaces.)

4.5 Space curves of constant curvature

All of our examples so far have had the property that, when the distribution was homogeneous, the rigid curves were also homogeneous. However, this is not generally true. We will now present a counterexample.

Consider the configuration space of the orthonormal frame bundle $\mathcal{F}$ of $\mathbb{E}^3$. Thus, a point of $\mathcal{F}$ is of the form

$$\mathbf{f} = (\mathbf{x};\ \mathbf{e}_1,\ \mathbf{e}_2,\ \mathbf{e}_3) \tag{4.19}$$

where $\mathbf{x}$ is a point of $\mathbb{E}^3$ and $\mathbf{e} = (\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is an (oriented) orthonormal triple. As usual, there are well-defined 1-forms $\omega_i = \mathbf{e}_i \cdot d\mathbf{x}$ and $\omega_{ij} = -\omega_{ji} = \mathbf{e}_i \cdot d\mathbf{e}_j$ which satisfy the equations

$$\begin{aligned} d\mathbf{x} &= \mathbf{e}_i\,\omega_i & d\omega_i &= -\omega_{ij}\wedge\omega_j \\ d\mathbf{e}_i &= \mathbf{e}_j\,\omega_{ji} & d\omega_{ij} &= -\omega_{ik}\wedge\omega_{kj} \end{aligned} \tag{4.20}$$

It is easy to see that any integral curve of the distribution defined on $\mathcal{F}$ by the Pfaffian equations

$$\omega_2 = \omega_3 = \omega_{31} = 0 \tag{4.21}$$

on which ω_1 and ω_{21} are non-zero is the Frenet frame of a smooth non-degenerate curve in $\mathbb{E}^3$.

Consider now the rank 2 distribution $\mathcal{D}$ defined on $\mathcal{F}$ by the Pfaffian equations

$$\omega_2 = \omega_3 = \omega_{31} = \omega_{21} - \omega_1 = 0\,. \tag{4.22}$$

The $\mathcal{D}$-curves on which ω_1 is non-zero are clearly the Frenet frames of space curves satisfying $\kappa \equiv 1$.

Now, it is easy to see that the ranks of $\mathcal{D}_1$ and $\mathcal{D}_2$ are 3 and 5 respectively. Thus, we have the conditions of Prop 4.2.

In this case, it is easy to identify $Q \subset T^*\mathcal{F}$ with $\mathcal{F}\times\mathbb{R}^4$ (with coordinates p_2, p_3, p_{31}, and p_{21} on the $\mathbb{R}^4$-factor) so that

$$\begin{aligned} \Psi &= d\big(p_2\,\omega_2 + p_3\,\omega_3 + p_{31}\,\omega_{31} + p_{21}\,(\omega_{21}-\omega_1)\big) \\ &= (dp_2 - p_3\,\omega_{32} + p_{21}\,\omega_{12})\wedge\omega_2 + (dp_3 - p_2\,\omega_{23} + p_{21}\,\omega_{13})\wedge\omega_3 \\ &\quad + (dp_{31} + p_3\,\omega_1 - p_{21}\,\omega_{23})\wedge\omega_{31} + (dp_{21} + p_2\,\omega_1)\wedge(\omega_{21}-\omega_1) \\ &\quad + p_{31}\,\omega_{23}\wedge\omega_{21}\,. \end{aligned} \tag{4.23}$$

Clearly, Q_1 is defined by the locus $p_{31} = 0$ and $Q_1^\bullet$ is defined by the relations $p_{31} = 0$ and $(p_3, p_{21}) \neq 0$. On this locus, the characteristic system of Ψ is defined by the rank 8 system

$$\begin{aligned} \omega_2 = \omega_3 = \omega_{31} = \omega_{21} - \omega_1 &= 0 \\ dp_2 - p_3\,\omega_{32} - p_{21}\,\omega_1 &= 0 \\ dp_3 + p_2\,\omega_{32} &= 0 \\ dp_{21} + p_2\,\omega_1 &= 0 \\ p_3\,\omega_1 + p_{21}\,\omega_{32} &= 0 \end{aligned} \tag{4.24}$$

There is an 8-parameter family of characteristic curves, but simultaneously scaling all of the p-variables by a constant will clearly preserve the system and carry each integral curve into another integral curve representing the same rigid $\mathcal{D}$-curve in $\mathcal{F}$. Thus, there is a 7-parameter family of rigid $\mathcal{D}$-curves. The above system can be integrated, up to a point, as follows:

First, note that the functions p_3p_{21} and $p_2^2 + p_3^2 + p_{21}^2$ are each a first integral of the above system.

Now, consider the case where $p_3p_{21} = 0$. On any characteristic curve in $Q_1^\bullet$, either p_3 vanishes identically or else p_{21} vanishes identically. If p_{21} is identically zero, then ω_1 must vanish identically (since p_3 cannot vanish), which forces the curve in $\mathcal{F}$ to satisfy $d\mathbf{x} = d\mathbf{e}_1 = 0$, i.e., it represents a frame spinning at a fixed base point about its first leg. On the other hand, if p_3 vanishes identically, then $\omega_{32} = 0$ and the curve in $\mathcal{F}$ satisfies $d\mathbf{x} = d\mathbf{e}_3 = 0$, i.e., it represents a curve with torsion $\tau = 0$, a plane curve. We already know that the plane curves with $\kappa = 1$ are simply the circles of radius 1.

Next, consider the case where $p_3p_{21} \neq 0$. Then, of course, p_{21} never vanishes. Set $\tau = -p_3/p_{21} \neq 0$ and $\sigma = p_2/p_{21}$, and the above equations imply that $\omega_{32} = \tau\,\omega_1$ (so that τ is, indeed, the torsion of the underlying space curve) as well as the differential equations

$$\begin{aligned} d\tau &= 2\tau\sigma\,\omega_1\,, \\ d\sigma &= (\sigma^2 - \tau^2 + 1)\,\omega_1\,. \end{aligned} \tag{4.25}$$

Along any solution curve, we may take $\omega_1 = ds$ where s is the element of arc length measured from a point where $|\tau|$ acheives a minimum.

Solving the above equations, it follows that a space curve with $\kappa \equiv 1$ is locally rigid among all space curves with $\kappa \equiv 1$ if and only if there is a constant λ satisfying $|\lambda| \leq 1$ so that the torsion is given by the formula

$$\tau(s) = \frac{\lambda}{\cos^2 s + \lambda^2\,\sin^2 s} \tag{4.26}$$

(where s denotes arc-length along the curve measured from a point where τ is a minimum). Note that, except when λ equals 0 or ± 1, these curves are not homogeneous, i.e., they are not the orbits of any 1-parameter subgroup of the Euclidean motions.

4.6 Non-rigidity

As a final closing remark, let us note that rigidity when $n > 2$ seems to be a rarer phenomenon. For example, it can be shown that, for the generic distribution of rank 3 in $\mathbb{R}^5$ or $\mathbb{R}^6$, there are no rigid curves, even though there are non-regular ones. The proof that the non-regular curves in these cases are not rigid is non-trivial and we will defer it to a later paper. We will content ourselves by simply giving a few examples.

First, consider the distribution $\mathcal{D}$ on $\mathbb{R}^5$ with coordinates (x, y^1, y^2, z^1, z^2) defined by the Pfaffian equations

$$dy^1 - z^1\,dx = dy^2 - z^2\,dx = 0\,. \tag{4.27}$$

Note that $\mathcal{D}_1$ is of rank 5 at every point.

It is easy to compute that every immersed non-regular $\mathcal{D}$-curve can be parametrized in the form $\gamma\colon [0, 1] \to \mathbb{R}^5$ where

$$\gamma(t) = (x_0, y_0^1, y_0^2, z_0^1 + rt\cos\theta, z_0^2 + rt\sin\theta) \tag{4.28}$$

for some constants x_0, y_0^i, z_0^i, $r > 0$, and θ. By applying transformations which preserve the distribution $\mathcal{D}$, such a γ can be transformed into the special case

$$\gamma_0(t) = (0, 0, 0, t, 0)\,. \tag{4.29}$$

However, this $\mathcal{D}$-curve is clearly not rigid since, for any function h on the interval [0, 1] such that it and its derivative vanish at the endpoints, the curve

$$\gamma_h(t) = \bigl(h'(t),\; t\,h'(t) - h(t),\; 0,\; t,\; 0\bigr) \tag{4.30}$$

is a non-trivial variation of γ_0 through $\mathcal{D}$-curves.

As another example, let **x** and **y** denote coordinates on two copies of $\mathbb{R}^3$ and consider the rank 3 distribution $\mathcal{D}$ defined on $\mathbb{R}^6 = \mathbb{R}^3 \times \mathbb{R}^3$ defined by the Pfaffian equations

$$d\mathbf{y} - \mathbf{x} \times d\mathbf{x} = 0\,. \tag{4.31}$$

It is easy to see that there is a 7-parameter family of non-regular $\mathcal{D}$-curves and that they are all of the form

$$\gamma(t) = \big(\mathbf{x}(t), \mathbf{y}(t)\big) = \big(\,\mathbf{x}_0 + t\,\mathbf{u},\ \mathbf{y}_0 + t\,(\mathbf{x}_0 \times \mathbf{u})\,\big) \tag{4.32}$$

for some constant vectors $\mathbf{x}_0$, $\mathbf{y}_0$, and $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^3$. By using symmetries and reparametrization, each of these curves can be brought into the form $\gamma_0\colon [0,1] \to \mathbb{R}^6$ given by

$$\gamma_0(t) = \big(\,t\,\mathbf{e}_1,\ \mathbf{0}\,\big), \tag{4.33}$$

so it suffices to show that this curve is not rigid. However, again, for any function h on the interval $[0,1]$ such that it and its derivative vanish at the endpoints, the curve

$$\gamma_h(t) = \big(\,t\,\mathbf{e}_1 + h'(t)\,\mathbf{e}_2,\ (t\,h'(t) - 2h(t))\,\mathbf{e}_3\,\big), \tag{4.34}$$

is a $\mathcal{D}$-curve with the same endpoints as γ_0. Thus, this curve is not rigid.

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