A Study of Singularity Formation in the Kelvin–Helmholtz Instability with Surface Tension

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Abstract

We consider the possibility of singularity formation in vortex sheets acted on by surface tension. In the evolution of a nearly flat vortex sheet with viscosity and surface tension neglected, Kelvin-Helmholtz instability has been shown to lead to the development of curvature singularities on the sheet. This is not unexpected, as the linearized equations of motion for this flow exhibit short wavelength instability: infinitesimal perturbations grow at an infinite rate. Since the inclusion of surface tension stabilizes (on linear theory) the high wavenumber modes, it has been presumed that it also suppresses the formation of singularities. We examine the possibility of singularity formation by generalizing Moore's approximation for vortex sheets to include the presence of surface tension. For the approximate theory, traveling wave solutions with imaginary wave speed are found which develop curvature singularities on the interface in finite time. Computations of the traveling wave solution are provided to reveal other features, and comparisons are made between these solutions and previously computed solutions of the full equations.

1 Introduction

Recent studies of the evolution of vortex sheets in two-dimensional, incompressible flows have shed much light on their behavior. The attention which vortex sheets have received is in part due to their importance as models of free shear layers in high Reynolds number flows. Examples of vortex sheet flows include those in which the vorticity is produced by a velocity discontinuity present initially in the flow, those in which the vorticity is generated baroclinically along a curve of density discontinuity, and those in which the vorticity is produced by a jump in viscosity across the interface. In each of these examples, an initial value problem can be prescribed in which the motion is subject to a well known instability: Kelvin-Helmholtz instability in the first case, Rayleigh-Taylor instability in the second, and Hele-Shaw instability in the the third.

When physical regularizations such as surface tension are absent, the form of these instabilities is such that the shortest wavelength components of a small perturbation grow the fastest [15]. Unfortunately, such behavior is disastrous from the standpoint of numerical computation, as mesh refinement leads to fast growing short wavelength modes which quickly swamp a calculation. If this pattern of growth persists into the non-linear regime, then the motion is ill-posed; small deviations in the initial condition will produce significant changes in the ensuing motion. In addition, singularities can form on the interface in finite time.

Singularity formation is one of the most interesting phenomenon for such free surface flows, and has received a lot of attention in the literature over the past 15 years (Caflisch [8] includes a brief review). These singularities, if they occur, are important to mathematical theory and numerical computation, while their physical significance depends on the particular problem. For example, in flows subject to the Kelvin-Helmholtz and Rayleigh-Taylor instabilities, the type of singularity may be related to the type of the subsequent roll-up of the sheet. This, in turn, determines the rate of mixing in the flow.

Recently, a new picture of singularity formation in vortex sheets has emerged by analytically extending the equations for the location of the interface into the complex domain. Specifically, let $(x(\alpha, t), y(\alpha, t))$ be a parametric representation of the interface at time t, and consider α to be a complex valued variable. This continuation into the complex plane is a mathematical procedure with no physical meaning; the physical interface corresponds to the restriction of x and y to real α. Nevertheless, singularities in x and y which may be present in the complex plane have a long range influence on the form of the physical interface. In addition, these singularities can move towards the real α axis. When one reaches this axis, it actually appears on the interface.

For periodic disturbances to a flat vortex sheet with non-zero mean vorticity, asymptotic studies by Moore [20], [19] and by Caflisch, Orellana, and Siegel [11] show how singularities in the complex α plane move towards and reach the real α axis in finite time, upon which a curvature singularity forms on the interface. These studies are based on a simple approximate theory for the evolution of vortex sheets using using a system of differential equations in the parameter α (here α has a physical interpretation as the circulation along the sheet). A major advantage of the approximate description is that it is local in α, as opposed to the exact description, which is given by a non-local integro-differential equation. Numerical calculations for the full equations by Meiron, Baker and Orszag [18], Krasny [17], and Shelley [25] verify most of the predictions of the asymptotic studies.

A similar set of approximate equations were derived for the Rayleigh-Taylor instability in [11]. Baker, Caflisch and Siegel [2] obtained a family of exact solutions to the approximate equations for which singularities develop on the fluid interface. The resulting predictions for the time, type and location of singularity formation were directly verified by numerical computations of the full equations, for a wide range of Atwood numbers (a measure of the density difference of the fluids) [2]. Numerical evidence for singularities in the Rayleigh-Taylor instability had been previously supplied by Pugh [22] for the Boussinesq limit. Tanveer [26] provided evidence that the single fluid case does not exhibit singularity formation. A related investigation of singularities for Boussinesq bubbles was performed by Cowley and Pugh [21].

The large body of numerical and analytical work cited above provides strong evidence that, for vortex sheet flows in the absence of regularization, singularities can develop in finite time. In order to prevent singularities from reaching the real axis, it is necessary to introduce some form of regularization. Numerical computations have been used to examine various regularizations (see, for example, [16], [27], and [3]). These computations have been able to proceed beyond the singularity time and show the formation of rolled-up vortex sheets. However, many of these regularizations do not correspond to a real physical effect (an exception is given in [3]).

One of the most prominent forces present in real interfacial flows is that of surface tension. This force stabilizes the high wavenumber modes in each of the Kelvin-Helmholtz, Rayleigh-Taylor, and Hele-Shaw instabilities. Since surface tension removes the ill-posedness in the linear theory, it has been suggested that it does also in the non-linear case [6]. However, we know of no published analytical study which has demonstrated this in the instabilities mentioned above. A recent numerical treatment by Hou, Lowengrub and Shelley [14] suggests that curvature singularities form precisely at the instant that the interface self-intersects. Other numerical studies are [23], [24], and [1].

In this paper, we address the issue of singularity formation in the Kelvin-Helmholtz instability with surface tension. This is done by generalizing Moore's approximation to include the effects of surface tension. As in [2], exact traveling wave solutions to the approximate equations are found which develop singularities in finite time. From these solutions, predictions are made concerning the type, time, and location of singularity formation. The predictions of the type of singularity are consistent with the results of Hou, Lowengrub, and Shelley (M. Shelley, private communication).

Our approximation is formally valid for small deformations of the interface and for singularities that are far from the real axis in the complex plane. In the present study, the theory is applied for large deformations and for singularities that are near the real axis, i.e., beyond the range of its formal validity. Nevertheless, the success of this approach in the previous study [2], in which the theory is also applied in a regime beyond its formal validity, motivates the present study.

The exact equations of motion for a vortex sheet acted on by gravity and surface tension are presented in §2. In §3, the system of localized equations is derived, and in §4 traveling wave solutions to this system are obtained. Computations of these solutions are performed in §5. In §6, we discuss several discrepancies between the traveling wave solutions and previously computed solutions to the full equations. Concluding remarks are presented in §8.

2 Governing Equations

Consider the two-dimensional motion of two immiscible, incompressible, inviscid fluids separated by a sharp interface. The subscripts 1 and 2 are used to refer to the lower and upper fluids, respectively. We assume that the two fluids have constant densities ρ_1 and ρ_2. The interfacial shape is assumed to be periodic in the horizontal direction with period 2π. It is further assumed that there is no circulation in the interior of each fluid, so that any vorticity as confined to be on the interface. Thus the interface is a vortex sheet.

It is convenient to use complex coordinates to specify the position of the interface. We let $z(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$ denote the position of the interface in the complex plane in terms of the time t and a parameter α that labels a particular marker particle on the sheet.

Define the velocities $\vec{u}_1$ and $\vec{u}_2$ as the limiting values of the fluid velocities as the interface is approached from below and above, respectively. Conservation of mass requires that they have equal normal components, i.e.,

$$\vec{u}_1 \cdot \vec{n} = \vec{u}_2 \cdot \vec{n}\,.$$

In the absence of viscosity, however, the tangential components can be discontinuous. The vorticity at a point $z(p)$ on the interface is due to a jump in the tangential velocity of the fluids near $z(p)$. We define the vortex sheet strength $\tilde{\gamma}$ by

$$\tilde{\gamma} = (\vec{u}_1 - \vec{u}_2) \cdot \widehat{s}$$

where $\widehat{s}$ is a unit vector tangent to the sheet at $z(p)$, pointing to the right relative to the direction from lower to upper fluid. It is also convenient to define the unnormalized vortex sheet strength γ by

$$\gamma = \tilde{\gamma}|z_\alpha|\,.$$

Although the normal component of the interfacial velocity $\vec{U}$ is required to satisfy $\vec{U} \cdot \vec{n} = \vec{u}_1 \cdot \vec{n} = \vec{u}_2 \cdot \vec{n}$, there is freedom in defining the tangential component. Following Baker et.al. [4], we choose $\vec{U}$ to be a weighted average of $\vec{u}_1$ and $\vec{u}_2$, i.e.,

$$\vec{U} = \frac{1+\beta}{2}\vec{u}_1 + \frac{1-\beta}{2}\vec{u}_2$$

where $\beta \in [-1,1]$. Note that by choosing $\beta = -1$ the Lagrangian motion follows the upper fluid, whereas for $\beta = 0$, this motion satisfies $\vec{U} = \frac{1}{2}(\vec{u}_1 + \vec{u}_2)$, the average velocity. In numerical simulations, this definition of the sheet velocity allows one to obtain some control over the positioning of the Lagrangian markers. We also find that a particular choice of β simppifies the analysis of §4.

Given the position and strength of the vortex sheet, its complex velocity $\partial\overline{z}/\partial t$ is prescribed by the following formula derived by Baker et. al. [4],

$$\overline{z}_t(\alpha,t) \;=\; \frac{1}{2\pi i}PV\int_{-\infty}^{\infty} \frac{\gamma(\alpha',t)}{z(\alpha,t) - z(\alpha',t)}d\alpha' + \frac{\beta\gamma(\alpha,t)}{2z_\alpha(\alpha,t)} \qquad (1)$$

where the integral is in the Cauchy principal value sense. Subscripts α and t refer to differentiation, and an overbar denotes complex conjugate. The integral term in (1) was first derived by Birkhoff and is known as the Birkhoff-Rott integral; it gives the mean of the two fluid velocities as a point $z(\alpha, t)$ on the interface is approached from either side. The second term on the right hand side of (1) results from our definition of $\vec{U}$ as a weighted average of $\vec{u}_1$ and $\vec{u}_2$.

In order to complete the description of the motion, an evolution equation for γ is required. For a vortex sheet which is acted upon by gravitational and surface tension forces, this equation is given by

$$\begin{aligned}\gamma_t(\alpha,t) &= 2A\left[Re\left\{\frac{\partial\overline{q}}{\partial t}z_\alpha\right\} - \frac{1}{2}\beta\gamma Re\left\{\frac{q_\alpha}{z_\alpha}\right\} + \frac{1}{8}\frac{\partial}{\partial\alpha}\,\frac{\gamma^2}{z_\alpha\overline{z}_\alpha} + gIm\{z_\alpha\}\right] \\ &+ \frac{1}{2}\beta\frac{\partial}{\partial\alpha}\,\frac{\gamma^2}{z_\alpha\overline{z}_\alpha} - \frac{2\tau}{(\rho_1+\rho_1)}\,\frac{\partial}{\partial\alpha}\left(\frac{1}{R}\right) \qquad (2)\end{aligned}$$

in which Re and Im denote the real and imaginary parts. Here A is the Atwood ratio, defined as

$$A = \frac{\rho_2 - \rho_1}{\rho_1 + \rho_2}\,,$$

g is the acceleration due to gravity, R is the radius of curvature at the point α on the sheet (defined as positive when fluid 1 protrudes into fluid 2), τ is the surface tension coefficient and $\overline{q}$ is the Birkhoff-Rott integral

$$\overline{q} = \frac{1}{2\pi i}PV\int_{-\infty}^{\infty}\frac{\gamma(\alpha',t)}{z(\alpha,t)-z(\alpha',t)}d\alpha'\,. \qquad (3)$$

Equation (2) is obtained by subtracting the tangential components of Euler's momentum equation across the interface. The difference between pressures on either side of the interface is expressed in terms of interface quantities using Laplaces equation $p_1 - p_2 = \tau/R$. A detailed derivation of equation (2) without the surface tension term is given by Baker, Meiron and Orszag [4]. Derivations wich include the surface tension term are provided by Tryggvason [27] and Pullin [23].

We comment briefly on the physical meaning of each of the terms in the vorticity evolution equation (2). The term

$$\left[Re\left\{\frac{\partial\overline{q}}{\partial t}z_\alpha\right\} + \frac{1}{8}\,\frac{\partial}{\partial\alpha}\,\frac{\gamma^2}{z_\alpha\overline{z}_\alpha}\right] \qquad (4)$$

describes the rate of change of the (unnormalized) vortex sheet strength following marker particles due to the differential tangential accelerations of the fluids above and below the interface [27]. If $\beta \neq 0$, the terms $\frac{1}{2}\beta\frac{\partial}{\partial\alpha}\frac{\gamma^2}{z_\alpha\bar{z}_\alpha}$ and $A\beta\gamma Re\left\{\frac{q_\alpha}{z_\alpha}\right\}$ are corrections to this rate due to the use of a weighted average for the interface velocity. The terms $2AgIm\{z_\alpha\}$ and $\frac{2\tau}{(\rho_1+\rho_2)}\frac{\partial}{\partial\alpha}\left(\frac{1}{R}\right)$ describe the rate of change of vorticity due to the action of gravity and surface tension, respectively. We note that the surface tension term $\frac{2\tau}{(\rho_1+\rho_2)}\frac{\partial}{\partial\alpha}\left(\frac{1}{R}\right)$ acts so as to oppose any interfacial growth or contortion [24]. This contrasts the situation in the break up of a 3-D cylindrical jet, in which the effect of surface tension is to induce the break up if the disturbance wavelength is larger than the jet circumference.

A simple solution to equations (1,2) is

$$z(\alpha,t) = \alpha + \frac{\beta\gamma_0}{2}t, \qquad \gamma(\alpha,t) = \gamma_0\,. \tag{5}$$

This solution corresponds to a flat interface with uniform vortex sheet strength γ_0, with the marker particles translating to the left with speed $\frac{\beta\gamma_0}{2}$. This would be an equilibrium solution in an Eulerian frame of reference.

The linearized version of equations (1,2) about the flat interface (5) permits solutions of the form

$$\begin{aligned} z(\alpha,t) &= \alpha + \frac{\beta\gamma_0}{2}t + Ae^{ik\alpha\pm\sigma(k)t} \\ \gamma(\alpha,t) &= \gamma_0 + Be^{ik\alpha\pm\sigma(k)t} \end{aligned}$$

where

$$\sigma(k) = \frac{ik(A+B)\gamma_0 \pm [(1-A^2)\gamma_0^2k^2 - 4\tau|k|^3 + 4\mathrm{Ag}|k|]^{1/2}}{2} \tag{6}$$

and, since we have assumed 2π periodic disturbances, k is an integer. There is an additional branch to the dispersion relation, $\sigma = 0$, which is usually termed a labeling mode and is an artifact of the Lagrangian approach (see [2]).

The dispersion relation has several features which are relevant to this study:

1. The choice $\beta = -A$ in (6) removes the imaginary part of σ; i.e., it removes the translational component of σ.

2. For $\gamma_0 \neq 0$ and $\tau = 0$, the real part of σ, which is the growth (or decay) rate of the linear modes, is proportional to $|k|$ for large k. This growth rate is an indication of Kelvin-Helmholtz instability. For $g \neq 0$ and $\gamma_0 = \tau = 0$, the real part of σ is proportional to $|k|^{\frac{1}{2}}$ for large k. This growth rate is an indication of Rayleigh-Taylor instability.

3. For $\gamma_0 \neq 0$ and $0 < \tau < \dfrac{(1-A^2)\gamma_0^2}{4} + Ag$, there are a finite but non-zero number of unstable modes. Above a cut-off wavenumber k_c, given by
$$k_c = \frac{(1-A^2)\gamma_0^2 + [(1-A^2)^2\gamma_0^4 + 32\tau Ag]^{1/2}}{8\tau},$$
each $\sigma(k)$ is pure imaginary. Thus interfacial surface tension has a stabilizing influence.

4. For $\gamma_0 \neq 0$ and $\tau \geq \dfrac{(1-A^2)\gamma_0^2}{4} + Ag$ every mode $\sigma(k)$ is pure imaginary, indicating complete suppression of the linearized instability for large surface tension.

Birkhoff [6] was the first to argue that the unbounded growth rate in the linearized dispersion relation for vortex sheets with $\gamma_0 \neq 0$, $\tau = 0$ allows the possibility of singularity development. Since then, analytic studies by Moore [20], [19], Caflish and Orellana [10] and Duchon and Robert [13], and numerical studies by Meiron, Baker and Orszag [18], Krasny [17], and Shelley [25] have all supported the conclusion that singularities form in finite time. If $\gamma_0 = 0$, Birkhoff's argument for singularity development based on linear analysis fails. Nevertheless, studies by Pugh [22] and Baker, Caflisch and Siegel [2] in the classical Rayleigh Taylor instability (i.e., for $\gamma_0 = 0$, $\tau = 0$ and $g \neq 0$) conclude that singularities do develop in finite time. With singularity development in the unregularized problem generally well established, we turn our attention to the question of singularity formation in the problem regularized (on linear theory) by surface tension. Under the action of an imposed shear and surface tension, the vortex sheet is subject to two competing tendencies: the tendency of the imposed shear to advect vorticity along the sheet toward an accumulation point leading to the build up of curvature, and the tendency of surface tension forces to avoid the build up of curvature. We develop a simple approximate theory which suggests that, despite the presence of surface tension, curvature singularities form.

3 The Localized Approximation

We shall derive a simple approximate theory for the evolution of vortex sheets acted on by surface tension and gravity. The approximate theory was first developed by Moore [19] in order to analyze singularity formation in the Kelvin-Helmholtz problem. It was later extended by Caflisch, Orellana and Siegel [11] to other interfacial flows, including the Rayleigh-Taylor instability, and by Caflisch [7] to smooth flows. For our purposes, we need only generalize the derivation given in [11] to include the surface tension term and the existence of a non-zero mean sheet vorticity. However, for completeness, we shall repeat the main points of the theory.

The approximation method is most easily described by extending the parameter α to be complex. Define an operation $*$ by

$$z^*(\alpha) = \overline{z}(\overline{\alpha})\,.$$

Note that z^* is the unique analytic function which equals $\overline{z}(\alpha)$ for α real. With this definition, the evolution equations (1,2) can be analytically extended to complex α as

$$\partial_t z^* = q^* + \frac{\beta\gamma}{2z_\alpha}$$

$$\begin{aligned}\partial_t\gamma \;=\; & A\Bigg[q_t^* z_\alpha + q_t z_\alpha^* - \frac{1}{2}\beta\gamma\left(\frac{q_\alpha}{z_\alpha} + \frac{q_\alpha^*}{z_\alpha^*}\right) \\ & -ig(z_\alpha - z_\alpha^*)\Bigg] + \frac{A+2\beta}{4}\,\frac{\partial}{\partial\alpha}\left(\frac{\gamma^2}{z_\alpha z_\alpha^*}\right) \\ & -\frac{2\tau}{(\rho_1+\rho_2)}\,\frac{\partial}{\partial\alpha}\mathrm{Im}\frac{z_{\alpha\alpha}^* z_\alpha}{(z_\alpha z_\alpha^*)^{3/2}}\end{aligned} \tag{7}$$

where the complex velocity q^* is given by

$$q^*(\alpha,t) \;=\; \frac{1}{2\pi i} PV \int_{-\infty}^{\infty} \frac{\gamma(\alpha+\xi,t)}{z(\alpha,t)-z(\alpha+\xi,t)}\, d\xi\,. \tag{8}$$

Knowledge of the behavior of the kernel in the above principal value integral (i.e., the positions and types of singularities, and behavior at infinity) can enable the integral to be evaluated by applying the residue theorem over a suitably deformed contour. This motivates a decomposition of the kernel into a sum of terms with known analytic behavior in the complex plane, plus a remainder term. To implement this kernel decomposition, we first decompose γ and z as

$$\begin{aligned} z(\alpha,t) &= \alpha + s_+(\alpha,t) + s_-(\alpha,t) + s_0(t) \\ \gamma(\alpha,t) &= \gamma_+(\alpha,t) + \gamma_-(\alpha,t) + \gamma_0(t) \end{aligned} \tag{9}$$

in which s_+, γ_+ contain the positive wavenumber components of s, γ respectively and s_-, γ_- contain the negative wavenumbers, i.e.,

$$\begin{aligned} s_+(\alpha,t) &= \sum_{k>0} \hat{s}_k(t) e^{ik\alpha} \\ s_-(\alpha,t) &= \sum_{k<0} \hat{s}_k(t) e^{ik\alpha} \\ \gamma_+(\alpha,t) &= \sum_{k>0} \hat{\gamma}_k(t) e^{ik\alpha} \\ \gamma_-(\alpha,t) &= \sum_{k<0} \hat{\gamma}_k(t) e^{ik\alpha}\,. \end{aligned}$$

If s and γ are analytic in the strip $|\mathrm{Im}\ \alpha| < \rho$, then s_+ and γ_+ are analytic in the upper half plane $\mathrm{Im}\ \alpha > -\rho$ and are called *upper analytic*. Similarly, s_- and γ_- are analytic in the lower half plane $\mathrm{Im}\ \alpha < \rho$ and are called *lower analytic*. The constant term in z, denoted by $s_0(t)$, plays no role in the resulting equations, since it does not effect the values of the Fourier coefficients $\hat{s}_k(t)$, $\hat{\gamma}_k(t)$ for $k \neq 0$. The constant term γ_0 in the representation of γ is the mean vortex sheet strength. Denote

$$z_{\alpha+} = 1 + s_{\alpha+} \qquad\qquad z_{\alpha-} = 1 + s_{\alpha-}\,. \tag{10}$$

Note that

1. The $*$-operator switches $+$ components to minus components, i.e.

$$(s^*)_+ = (s_-)^*\,, \qquad (s^*)_- = (s_+)^*\,.$$

2. Since γ is real when α is real, $\hat{\gamma}_{-k} = \overline{\hat{\gamma}}_k$ so that

$$(\gamma_+)^* = \gamma_-\,.$$

The analytically extended equations ( ) can be written as

$$\begin{aligned} z_t^* &= B_1[z_\alpha, z_\alpha^*, \gamma, \gamma^*] \\ \gamma_t &= B_2[z_\alpha, z_\alpha^*, \gamma, \gamma^*]\,. \end{aligned}$$

For any function as operation $A[s]$, such as the kernel in the principal value integral (8) or any other terms in (7), we can write

$$A[s] = A[s_+] + A[s_-] + E[s_+, s_-]\,.$$

This is just a definition of the term E. The approximation takes the form

$$A[s] \sim A[s_+] + A[s_-]\,.$$

If s has a constant term, s_0, the corresponding approximation is

$$A[s] \sim A[s_+ + s_0] + A[s_- + s_0] - A[s_0]\,.$$

The neglected term E contains only cross products of upper and lower analytic functions such as s_+s_-. Pure products such as s_+^2 or s_-^2 are contained in $A[s_+]$ and $A[s_-]$. Thus the approximation amounts to neglecting all cross products of upper and lower analytic functions. It is shown in [11] that the size of a typical neglected term of the form s_+s_- satisfies

$$\|s_+s_-\| \leq e^{-2\rho}\|s_+\|\;\|s_-\|$$

where $\|s\| = \sup_{|Im\ \alpha|\leq\rho} |s(\alpha)|$. For $\rho >> 1$, this is negligible.

The approximation is based on the following physical argument: Nonlinear interactions cause wavenumbers to add so that energy flows back and forth among wavenumbers. However, the process of singularity development primarily involves flow of energy from low wavenumbers to high wavenumbers. Our approximation keeps this flow of energy to higher wavenumbers, but ignores all flow of energy from high to low wavenumbers, i.e., all interactions between positive wavenumbers and negative wavenumbers are neglected. Mathematically, this is the same as keeping interactions of $+$ terms with $+$ terms and $-$ terms with $-$ terms, but neglecting interactions of $+$ with $-$.

In the present situation, we apply the approximation for both s and γ. Therefore, (7) becomes

$$\begin{aligned} z_t^* &= B_1(z_{\alpha+},(z^*)_{\alpha+},\gamma_+,(\gamma^*)_+) \\ &\quad +B_1(z_{\alpha-},(z^*)_{\alpha-},\gamma_-,(\gamma^*)_-) \\ \gamma_t &= B_2(z_{\alpha+},(z^*)_{\alpha+},\gamma_+,(\gamma^*)_+) \\ &\quad +B_2(z_{\alpha-},(z^*)_{\alpha-},\gamma_-,(\gamma^*)_-)\,. \end{aligned} \tag{11}$$

For example, in the second equation of (7), the surface tension term is approximated as

$$\frac{2\tau}{(\rho_1+\rho_2)}\frac{\partial}{\partial\alpha}\mathrm{Im}\frac{z^*_{\alpha\alpha}z_\alpha}{(z_\alpha z^*_\alpha)^{3/2}} \sim \frac{\tau}{i(\rho_1+\rho_2)}\,\frac{\partial}{\partial\alpha}\left[\frac{s^*_{\alpha\alpha-}s_{\alpha+}+s^*_{\alpha\alpha-}-s_{\alpha\alpha+}s^*_{\alpha-}-s_{\alpha\alpha+}}{(1+s_{\alpha+}+s^*_{\alpha-}+s_{\alpha+}s^*_{\alpha-})^{3/2}}\right.$$
$$\left.+\frac{s^*_{\alpha\alpha+}s_{\alpha-}+s^*_{\alpha\alpha+}-s_{\alpha\alpha-}s^*_{\alpha+}-s_{\alpha\alpha-}}{(1+s_{\alpha-}+s^*_{\alpha+}+s_{\alpha-}s^*_{\alpha+})^{3/2}}\right]$$

where we have used (10). Each of the other terms in (7) is similarly approximated.

An advantage of this approximation is that now the Birkhoff-Rott integral $q^*[s,\gamma]$ can be evaluated by integration along a suitably deformed contour, resulting in a purely local term coming from a residue. By using this evaluation, then projecting equations (11) onto upper and lower analytic function space, we obtain the following system:

$$\frac{\partial s^*_-}{\partial t} = \frac{\beta-1}{2}\left(\frac{\gamma_0+\gamma_+}{1+s_{\alpha+}}\right) \tag{12}$$

$$\frac{\partial s_+}{\partial t} = \frac{\beta+1}{2}\left(\frac{\gamma_0+\gamma_+}{1+s^*_{\alpha-}}\right) \tag{13}$$

$$\frac{\partial \gamma_+}{\partial t} = \frac{A+\beta}{2}\,\frac{\partial}{\partial\alpha}\left(\frac{(\gamma_0+\gamma_+)^2}{(1+s_{\alpha+})(1+s^*_{\alpha-})}\right) - i\mathrm{Ag}(s_{\alpha+}-s^*_{\alpha-}) \tag{14}$$

$$+\frac{i\tau}{(\rho_1+\rho_2)}\,\frac{\partial}{\partial\alpha}\left[\frac{s^*_{\alpha\alpha-}s_{\alpha+}+s^*_{\alpha\alpha-}-s_{\alpha\alpha+}s^*_{\alpha-}s_{\alpha\alpha+}}{(1+s_{\alpha+}+s^*_{\alpha-}+s_{\alpha+}s^*_{\alpha-})^{3/2}}\right]. \tag{15}$$

In obtaining the above, we have used the relation $(\gamma^*)_- = \gamma_+$, and have applied the $*$ operator to go from an equation for $\dfrac{\partial s_+^*}{\partial t}$ to one for $\dfrac{\partial s_+}{\partial t}$. For details of the localized approximation procedure and its application in similar problems, see [11].

It is convenient to write (15) as a quasi-linear system. Denote

$$\phi = 1 + s_{\alpha +} \qquad \psi = 1 + s^*_{\alpha -} \qquad \omega = \gamma_+ \tag{16}$$

and differentiate to first two equations of (15) to obtain

$$\begin{aligned} \phi_t &= \frac{\beta + 1}{2} \frac{\partial}{\partial \alpha} \left(\frac{\omega + \gamma_0}{\psi} \right) \\ \psi_t &= \frac{\beta - 1}{2} \frac{\partial}{\partial \alpha} \left(\frac{\omega + \gamma_0}{\phi} \right) \end{aligned}$$

$$\begin{aligned} \omega_t = &\frac{A + \beta}{2} \frac{\partial}{\partial \alpha} \left(\frac{(\omega + \gamma_0)^2}{\phi \psi} \right) - i\mathrm{Ag}(\phi - \psi) \\ &+ \frac{i\tau}{(\rho_1 + \rho_2)} \frac{\partial}{\partial \alpha} \left[\frac{\psi_\alpha \phi - \phi_\alpha \psi}{(\phi \psi)^{3/2}} \right] . \end{aligned} \tag{17}$$

The localized equations are much simpler than the full equations (1,2), mainly in that they are purely local differential equations rather than non-local integro-differential equations. However, as described in [2], there is a defect in the approximate system concerning the role of β. For the full equations, solutions obtained for different values of β are equivalent. The parameterization of the solution in α depends on the particular value of β, but the resulting curve does not. This is not the case for the approximate system, since different values of β lead to different approximations. The reason for this is simple. Different choices of β correspond to different parameterizations by α for the motion. The approximation involves Fourier transformation to make the decompositions into upper and lower analytic functions. Since changes of variables are not easily expressed through the Fourier transform, there is no simple relation between solutions of the approximate system for different choices of β. Nevertheless, using an argument analogous to that given in Caflisch and Orellana [9] or Baker, Cowley, Tanveer and Page [12], one can establish that solutions to the approximate system for various β are asymptotically close to the exact solution when the deformations of the interface are small and the singularities are far from the real axis. Consequently, the nature of the singularities are predicted reliably by the approximate system independently of β.

We next consider special solutions of the localized equations (17) which take the form of traveling wave solutions in the complex plane. These solutions will be shown to have singularities which move with constant speed toward the real α axis. The singularities reach this axis at a time $t = t_c$, upon which a curvature singularity appears on the fluid interface.

4 Traveling Wave Solutions

We seek solutions to the localized equations (17) that are 2π periodic in space. Note that these equations contain only upper analytic variables, namely ϕ, ψ and ω. The only other coefficients in the system are constants, i.e., there are no coefficients or terms which are known functions of α and t. An unusual property of such systems (or analogous lower analytic systems) is that they often allow exact solutions which take the form of periodic traveling waves in the complex plane. Furthermore, the speed of a given traveling wave solution is independent of the wave amplitude and identical to the speed given by a linear analysis. This contrasts the behavior of traveling wave solutions to general non-linear PDE's, which have speeds that typically depend on the amplitude of the wave.

The existence of such traveling wave solutions in upper analytic PDE's can be illustrated by considering a single quasi-linear first order PDE in α and t of the form

$$f_{t+} + L[f_+]f_{\alpha+} \quad = \quad R[f_+] \tag{18}$$

where f_+ is an upper analytic variable. It is assumed that $L[f_+]$ and $R[f_+]$ admit a Taylor's series expansion in f_+ about some neighborhood of $f_+ = 0$, and that the coefficients of the Taylor's expansion are constants independent of α and t. Furthermore, it is assumed that $R[0] = 0$. We then look for solutions of the form

$$f_+ \quad = \quad a_1 e^{\xi} + a_2 e^{2\xi} + \cdots \quad \text{with} \quad \xi = i(\alpha + \sigma t)\,, \tag{19}$$

where the a_i's are constants. This is just a Fourier expansion of a periodic traveling wave with speed σ. The absence of a term a_0 is justified by the assumption $R[0] = 0$. Substituting this form into (18) and equating coefficients to like powers of e^ξ leads to the following hierarchy of equations

$$\begin{aligned} a_1 F_1(\sigma) &= 0 \\ F_2(\sigma; a_1, a_2) &= 0 \\ &\vdots \\ F_j(\sigma; a_1, a_2, \ldots, a_j) &= 0 \\ &\vdots \end{aligned} \tag{20}$$

The F_i's are functions of the indicated variables. For the first equation to have non-trivial solutions with $a_1 \neq 0$, σ must satisfy $F_1(\sigma) = 0$. Thus σ is determined in the first (linear) order of the hierarchy. The remaining equations successively determine a_j as a function of $a_{j-1}, \ldots, a_1; \sigma$. The fact that (18) is a purely upper analytic equation and allows an outflow of energy to high wavenumbers while avoiding inward flow is crucial in producing the structure of this hierarchy, and results in the solution (19,20). Note that the traveling wave solution to (18) will be uniquely determined if we require that $f_+ \sim a_1 e^\xi$ as $\xi \to -\infty$ for a given a_1. We also remark that the existence of these special solutions makes the localized approximation a useful procedure, even for operators which are already local in form (see for example [7]).

Traveling wave solutions with non-zero imaginary component of speed are particularly significant with regard to singularity formation. For these solutions, singularities travel in the complex plane toward the real α line with speed $\mathrm{Im}(\sigma)$. A singularity $\alpha_s = x_s + iy_s$, with imaginary component y_s at $t = 0$, will hit the real α line at $t = \dfrac{y_s}{\mathrm{Im}(\sigma)}$. Consequently a singularity in the traveling wave solution will occur on the interface in finite time, if the initial data has a singularity at any complex value of α.

Motivated by the above remarks, we look for solutions to the localized equations of the form $\phi = \phi(\xi)$, $\psi = \psi(\xi)$ and $\omega = \omega(\xi)$ with $\xi = i(\alpha + \sigma t)$. We require that $\phi \sim 1 + \widehat{\phi}_1 e^\xi$, $\psi \sim 1 + \widehat{\psi}_1 e^\xi$ and $\omega \sim \widehat{\omega}_1 e^\xi$ as $\xi \to -\infty$, and expect that σ will be given by

$$\sigma = \frac{(A + \beta)\gamma_0}{2} \pm i \left[\frac{(1 - A^2)}{4} \gamma_0^2 - \tau + Ag \right]^{1/2} \tag{21}$$

which is identical to the speed obtained from a linear analysis (this must be verified at the first order of the hierarchy described earlier). Substitution of these forms into (17) gives the following system of ODE's:

$$\begin{aligned}
\sigma\phi_\xi &= \frac{\beta+1}{2}\frac{\partial}{\partial\xi}\left(\frac{\omega+\gamma_0}{\psi}\right) \\
\sigma\psi_\xi &= \frac{\beta-1}{2}\frac{\partial}{\partial\xi}\left(\frac{\omega+\gamma_0}{\phi}\right) \\
\sigma\omega_\xi &= \frac{A+\beta}{2}\frac{\partial}{\partial\xi}\left(\frac{(\omega+\gamma_0)^2}{\phi\psi}\right) - \mathrm{Ag}(\phi-\psi) \\
&\quad -\frac{\tau}{(\rho_1+\rho_2)}\frac{\partial}{\partial\xi}\left[\frac{\psi_\xi\phi-\phi_\xi\psi}{(\phi\psi)^{3/2}}\right]\,.
\end{aligned} \tag{22}$$

The first two equations can be integrated directly, yielding

$$\begin{aligned}
\phi &= \phi(0)+\frac{\beta+1}{2\sigma}\left(\frac{\omega+\gamma_0}{\psi}-\frac{\omega(0)+\gamma_0}{\psi(0)}\right) \\
\psi &= \psi(0)+\frac{\beta-1}{2\sigma}\left(\frac{\omega+\gamma_0}{\phi}-\frac{\omega(0)+\gamma_0}{\phi(0)}\right)\,.
\end{aligned} \tag{23}$$

By the definitions (16), the constant part in the Fourier expansions for both ϕ and ψ is equal to one, whereas there is no constant part in the Fourier expansion of ω. Consequently, the right hand sides of (23) must have a mean value of 1, and this implies that the following relationships must hold:

$$\begin{aligned}
\phi(0)+\frac{\beta+1}{2\sigma}\left(\gamma_0-\frac{\omega(0)+\gamma_0}{\psi(0)}\right) &= 1 \\
\psi(0)+\frac{\beta-1}{2\sigma}\left(\gamma_0-\frac{\omega(0)+\gamma_0}{\phi(0)}\right) &= 1\,.
\end{aligned} \tag{24}$$

Define

$$\begin{aligned}
\Phi_0 &= 1-\frac{\beta+1}{2\sigma}\gamma_0 \\
\Psi_0 &= 1-\frac{\beta-1}{2\sigma}\gamma_0\,.
\end{aligned} \tag{25}$$

Substitution of the relations (24) and (25) into (23) then gives

$$\begin{aligned}
\phi &= \Phi_0 + \frac{1+\beta}{2\sigma}\,\frac{\omega+\gamma_0}{\psi} \\
\psi &= \Psi_0 - \frac{1-\beta}{2\sigma}\,\frac{\omega+\gamma_0}{\phi}\,.
\end{aligned} \tag{26}$$

It is possible to express the third equation in (22) as an ODE with a single dependent variable. A convenient choice for the dependent variable is $\Omega = \dfrac{\omega+\gamma_0}{\sigma\Phi_0\Psi_0}$. We use equations (26) to express ϕ and ψ in terms of Ω:

$$\begin{aligned}
2\phi &= \Phi_0(1+\Omega \pm \sqrt{1+2\beta\Omega+\Omega^2}) \\
2\psi &= \Psi_0(1-\Omega \pm \sqrt{1+2\beta\Omega+\Omega^2})\,.
\end{aligned} \tag{27}$$

Here the square root is defined using the principal branch cut, i.e. $(1+2\beta\Omega+\Omega^2)^{1/2} = \sqrt{M}e^{i\theta/2}$, where $\theta \in [-\pi,\pi]$ is the argument of $1+2\beta\Omega+\Omega^2$, and M is its amplitude. This choice of branch makes sense only if the curve $\mathcal{C}$ traced out by $1+2\beta\Omega+\Omega^2$ as α goes from 0 to 2π does not intersect the negative real axis. This is the case for $\Omega' = \Omega - \Gamma_0$ small, where

$$\Gamma_0 = \frac{\gamma_0}{\sigma\Phi_0\Psi_0} \tag{28}$$

is the mean of Ω. The situation is more complicated when $\mathcal{C}$ intersects the negative real axis; this will be discussed momentarily. The sign in front of the square root is selected by the requirement that the constant parts in the Fourier expansions of ϕ and ψ equal one. This requirement can lead to selection of either the $+$ or $-$ branch, depending on the values of the parameters g, γ_0 and τ. The branch of the square root which is to be selected in a specific case will be pointed out when that case is discussed in detail. For now, we leave the $\pm$ in order to indicate this indeterminacy.

Upon substituting (27) into the third equation of (22) and manipulating some, we obtain

$$\begin{aligned}
\frac{\partial}{\partial\xi}\ &\left\{\ \frac{(A+\beta)\sigma^2\Omega^2}{v(\Omega)^2} - \sigma^2\Omega + \left(\frac{2}{\Phi_0^3\Psi_0^3}\right)^{\frac{1}{2}}\frac{\tau}{(\rho_1+\rho_2)}\frac{u(\Omega)}{v(\Omega)}\Omega_\xi\right\} \\
&= \frac{Ag}{2}\left[\left(\frac{1}{\Psi_0}-\frac{1}{\Phi_0}\right)(1+u(\Omega)) + \Omega\left(\frac{1}{\Psi_0}+\frac{1}{\Phi_0}\right)\right].
\end{aligned} \tag{29}$$

where

$$u(\Omega) = \pm(1 + 2\beta\Omega + \Omega^2)^{\frac{1}{2}},\ v(\Omega) = (1 + \beta\Omega \pm u(\Omega))^{\frac{1}{2}}. \tag{30}$$

Several situations lead to a considerable simplification of the above equation. As we have pointed out earlier, the choice $\beta = -A$ selects a convenient parameterization, and we see here that this choice eliminates the first term in (29). The case $\tau = 0$, $\gamma_0 = 0$, also leads to a marked simplification of this equation. This case corresponds to an interface subjected to a pure Rayleigh-Taylor instability and is treated in detail in [2]. Since this is a particularly simple situation, yet previews some of the features of the more difficult case $\tau \neq 0$, we briefly review the results of [2] here. For $\gamma_0 = 0$, Φ_0 and Ψ_0 are both equal to 1, and it is necessary to choose the + branch in (27,30) to satisfy the requirement that ϕ and ψ have a constant Fourier component equal to 1. If β is chosen to be $-A$, then equation (29) reduces to

$$\Omega_\xi = -\left(\frac{\mathrm{Ag}}{\sigma^2}\right)\Omega\,.$$

The requirement that Ω is upper analytic and 2π periodic in α forces $\frac{\mathrm{Ag}}{\sigma^2} = -n$ for integer n. A unique solution can be chosen by specifying the behavior of Ω as $\xi \to -\infty$. If we select $\Omega \sim \Omega(0)e^\xi$ as $\xi \to -\infty$, then $n = 1$ and the resulting solution is

$$\Omega = \Omega(0)e^\xi \qquad\qquad \sigma = \pm i(\mathrm{Ag})^{1/2} \tag{31}$$

where $\Omega(0)$ is a complex amplitude. Note that the speed σ is as expected from our linear analysis.

The traveling wave solutions correspond to a special choice of initial condition for the sheet position and sheet strength. For these initial conditions, the nature of the singularity is easily deduced from the explicit solutions (27), (31). In particular, for $\Omega(0)$ small, the functions ϕ and ψ have square root singularities in the lower half complex α plane which move vertically upwards with constant speed. These singularities reach the real α axis at positions

$$\alpha_c = \pi \pm \tan^{-1}\left(\frac{[1 - A^2]^{\frac{1}{2}}}{A}\right), \tag{32}$$

and at time

$$t_c = \frac{1}{(Ag)^{\frac{1}{2}}} \ln \frac{1}{|\Omega(0)|} \tag{33}$$

at which point they are physically realized. The values for the time and position of singularity formation obtained from the traveling wave solution were directly verified by numerical computation of the full equations, with excellent agreement between the theoretical preductions and numerical results for values of Atwood number between 0 and approximately .9. Further details, including traveling wave solutions for other values of β, are given in [2].

The predictions in the $\tau = 0$, $\gamma_0 = 0$ case agree well with the results of numerical calculations despite the fact that the approximate theory is applied for large deformations and for singularities that are near to the real axis, which is beyond its range of formal validity. The agreement is suggestive and motivates us to use the localized approximation theory to examine singularity formation in the $\tau \neq 0$ case.

4.1 Singularity Formation for $\tau \neq 0$

Our main objective is to consider the case $g = 0$, $\gamma_0 \neq 0$, $\tau \neq 0$; this is the simplest case in which $\tau \neq 0$ and which still exhibits unstable modes in the linear analysis (for sufficiently small τ). These parameter values correspond to a fluid interface which is subjected to a shear flow and, therefore, susceptible to the Kelvin-Helmholtz instability, but which is also acted on by the stabilizing influence of surface tension. For $g = 0$ the term on the right hand side of (29) is zero and consequently this equation can be directly integrated with respect to ξ. After performing this integration and solving for Ω_ξ, we obtain

$$\Omega_\xi = \kappa u(\Omega) v(\Omega) \left\{ \sigma^2 \Omega - \frac{(A+\beta)\sigma^2\Omega^2}{v(\Omega)^2} + C \right\} \tag{34}$$

where $\kappa = \dfrac{(\rho_1 + \rho_2)}{\sqrt{2}\tau} (\Phi_0 \Phi_0)^{3/2}$ and C is a constant of integration. Since Ω_ξ has zero mean (i.e., its constant Fourier mode is zero), the constant C is chosen so that the sum contained within the brackets in (34) has zero mean. This gives

$$C = -\sigma^2 \Gamma_0 + (A+\beta) \left[\frac{\sigma^2 \Gamma_0^2}{(1 + \beta\Gamma_0 \pm \sqrt{1 + 2\beta\Gamma_0 + \Gamma_0^2}} \right] \tag{35}$$

using definition (28).

We now examine the possibility of singularities in the traveling wave solution $\phi(\xi)$, $\psi(\xi)$, $\Omega(\xi)$. The differential equation for $\Omega(\xi)$ is simplest when $\beta = -A$, so we restrict ourselves to this case. Analysis of (34) for $\beta \neq -A$ leads to similar conclusions as for $\beta = -A$. With the simplification, (34) becomes

$$\Omega_\xi = F(\Omega) \tag{36}$$

where

$$F(\Omega) = \kappa u(\Omega) v(\Omega)\{\sigma^2 \Omega + C\} \tag{37}$$

We also require that $|\beta| = |A| \neq 1$, i.e., the interface separates two fluids of non-zero density. For certain values of A we have been able to integrate (36) and obtain the solution for implicit form. However, the implicit equation is complicated and it is simpler and perhaps more illuminating to analyze singularity formation without explicitly integrating. Therefore, we proceed with this indirect approach.

The requirement that ϕ and ψ have mean 1 now makes it necessary to choose the $(-)$ branch in (27,30). Since Ω is 2π periodic, upper analytic and has mean Γ_0, we can write $\Omega = \Gamma_0 + \widehat{\Omega}_1 e^\xi + O(e^{2\xi})$ for constant $\widehat{\Omega}_1$ and $\xi \to -\infty$. Substituting this into equation (36) and equating coefficients of the leading order term e^ξ, gives

$$\widehat{\Omega}_1 = \sigma^2 D \widehat{\Omega}_1$$

where $D = -\kappa(1 + 2\beta\Gamma_0 + \Gamma_0^2)^{1/2}[1 + \beta\gamma_0 - (1 + 2\beta\Gamma_0 + \Gamma_0^2)^{1/2}]^{1/2}$ is the mean of the term $u(\Omega)v(\Omega)$ on the right hand side of (36). The relation $\sigma^2 D = 1$ determines σ, and a straightforward but lengthy computation verifies that the result is given by (21), as expected. The solution to (36) is then uniquely determined by specifying $\widehat{\Omega}_1 = \lim_{\xi \to -\infty} \dfrac{\Omega - \Gamma_0}{e^\xi}$. Each coefficient $\widehat{\Omega}_n$ for $n > 1$ is determined from $\widehat{\Omega}_1, \ldots, \widehat{\Omega}_{n-1}$ and σ as described earlier.

Singularities can occur in $\Omega(\xi)$ when $F(\Omega) = 0$ or when $F(\Omega) = \infty$. Corresponding singularities in ϕ and ψ occur when $\Omega(\xi)$ is singular or when $1 + 2\beta\Omega + \Omega^2 = 0$ (this is also a situation in which $F(\Omega) = 0$). We now examine each of these possibilities:

1. $F(\Omega) = 0$: There are three situations which lead to $F(\Omega) = 0$. One is $u(\Omega) = 0$, a second is $v(\Omega) = 0$, and a third is given by $\Omega - \Gamma_0 = 0$. It can be shown that each of these situations does not produce a finite time

singularity in $\Omega(\xi)$. In the first situation, the function $\Omega(\xi)$ is analytic in a neighborhood of a point ξ_p where $u(\Omega) = 0$. To see this, rewrite (36) as a differential equation in the function $u = (1 + 2\beta\Omega + \Omega^2)^{1/2}$. This results in an equation of the form $u_\xi = q(u)$, where $q(u)$ is given by

$$q(u) = -[\kappa\sigma^2(\Omega(u) - \Gamma_0)][(\beta + \Omega(u))(1 + \beta\Omega(u) - u)^{1/2}] .$$

In the above, $\Omega(u) = -\beta + \sqrt{\beta^2 - 1 + u^2}$; the branch of the square root is not important for our purposes. The function $q(u)$ is analytic in a neighborhood of $u = 0$. Furthermore, $q(u = 0) \neq 0$ for $|\beta| \neq 1$, since the relation $1 + 2\beta\Omega + \Omega^2 = 0$ guarantees that $\Omega(u = 0) \neq -\beta$ and $1 + \beta\Omega \neq 0$, and we assume that $\Omega(u = 0) \neq \Gamma_0$ (the case $\Omega(u) = \Gamma_0$ is treated separately). Consequently, $\xi = \displaystyle\int \frac{1}{q(u)} du = h(u)$ is an analytic function of u in a neighborhood of $u = 0$. Since $\dfrac{dh}{du}(u = 0) \neq 0$, the function $\xi = h(u)$ can be locally inverted (for u near 0) to obtain $u = h^{-1}(\xi)$, which is analytic in a neighborhood of $\xi = h(0)$. Although $1 + 2\beta\Omega + \Omega^2 = 0$ can occur in finite time, the function $\Omega(\xi)$ remains analytic in a neighborhood of the point where this occurs.

The second of the three situations does not occur in finite time. This is demonstrated by writing (36) as a differential equation in the variable $v = (1 + \beta\Omega + u(\Omega))^{\frac{1}{2}}$, which leads to an equation that behaves like $v_\xi \sim cv$ for constant c and v near zero. This implies that $v = 0$, or equivalently $1 + \beta\Omega + u(\Omega) = 0$, only for infinite values of ξ.

Finally, we consider the possibility that $\Omega - \Gamma_0 = 0$. Writing (36) as a differential equation in the function $\Omega' = \Omega - \Gamma_0$ leads to an equation which has the form $\Omega'_\xi = c_1\Omega'$ for Ω' near 0, where c_1 is a constant. Again we conclude that $\Omega' = 0$ only for infinite values of ξ.

Singularities can still at points where $F(\Omega) = \infty$. We consider this possibility next.

2. $F(\Omega) = \infty$: This can only occur if $\Omega \to \infty$. For Ω near ∞,

$$F(\Omega) \sim d\ \Omega^{5/2}$$

where d is a constant which depends on the particular choice of branch in (36) but which is not zero. Thus equation (36) allows singularities of the form $\Omega \sim d_1(\xi - \xi_0)^{-2/3}$, and leads to singular behavior in z_ξ given by $z_\xi \sim d_2(\xi - \xi_0)^{-2/3}$, where d_1 and d_2 are constants.

It is necessary to verify that the singular behavior in Ω actually corresponds to a geometric singularity in the interfacial curve and does not just indicate a singularity in the parameterization used to represent the curve. This is most easily done by computing the sheet curvature $\mathcal{K} = \mathrm{Im}\left[\frac{z_\alpha z^*_{\alpha\alpha}}{(z_\alpha z^*_\alpha)^{3/2}}\right]$ near a point ξ_0 where $\Omega(\xi_0) = \infty$. Near such a point we find that $\mathcal{K} \sim b\,\Omega^{1/2}$, which indicates that the curvature has a singularity of the form $\mathcal{K} \sim b_1(\xi - \xi_0)^{-1/3}$, for constant b, b_1. The unnormalized vortex sheet strength $\gamma = \sigma\Phi_0\Psi_0(\Omega + \Omega^*)$ blows up at ξ_0; however the true vortex sheet strength $\tilde{\gamma} = \frac{\gamma}{|z_\alpha|}$ remains bounded.

In summary, for $A \neq 1$ and $\beta = -A$ singularities arise in the traveling wave solution at positions in the finite complex plane where $\Omega = \infty$. The singularities correspond to infinite curvature at the interface. Since $\Omega(\xi)$ is an upper analytic function, these singularities are located in the lower half of the complex α plane. The theory predicts that they move toward the real axis with constant speed σ, as long as $\tau < (1 - A^2)\gamma^2/4$ (required for σ to have a non-zero imaginary part). When they reach the real axis, the singularities appear on the sheet.

We remark that our results are consistent with those obtained by Baker, Cowley, Tanveer and Page [12]. They employ a local analysis similar to that which led to the system of PDE's (17), and show that curvature singularities in the complex plane of the form $\mathcal{K} \sim c(\xi - \xi_0)^{-1/3}$ are allowed by their localized equation.

The information which we now have concerning the analytic behavior of $\Omega(\xi)$ near a point where $u(\Omega) = \sqrt{1 + 2\beta\Omega + \Omega^2} = 0$ forces us to be more precise in defining a branch for the square root. Let the curve which is traced by $1 + 2\beta\Omega + \Omega^2$ as α ranges from 0 to 2π be denoted by $\mathcal{C}$. For $\Omega' = \Omega - \Gamma_0$ small in the range $\alpha = 0$ to 2π, i.e. for $\widehat{\Omega}_1 e^{\sigma t}$ small, the curve $\mathcal{C}$ will not cross the negative real axis.

As $\widehat{\Omega}_1 e^{\sigma t}$ (and therefore Ω') increases, the curve $\mathcal{C}$ eventually goes through zero; however, we have seen that $\sqrt{1 + 2\beta\Omega + \Omega^2}$ is not singular when this occurs. For larger values of $\widehat{\Omega}_1 e^{\sigma t}$ the curve $\mathcal{C}$ crosses the negative real axis, but remains an analytic function of ξ (see Figure 1). We define a branch of the square root which maintains analyticity of $(1 + 2\beta\Omega + \Omega^2)^{1/2}$ as a function of ξ when $\mathcal{C}$ crosses the negative real axis, in the following manner. Define $1 + 2\beta\Omega + \Omega^2 = Me^{i\theta}$ where M is the amplitude and θ is the angle of $1 + 2\beta\Omega + \Omega^2$ with respect to the x-axis. The angle $\theta = 0$ is defined so as to correspond to the right most crossing of the curve $\mathcal{C}$ with the x-axis, and the other values of θ are fixed by the requirement that θ be a continuous function of the arclength of $\mathcal{C}$. For example, in Figure 1, point A corresponds to $\theta = 0$, point B to an angle θ just greater than π, point C to $\theta = 2\pi$, and point D to an angle just greater than 3π. The square root is then defined by $(1 + 2\beta\Omega + \Omega^2)^{1/2} = \sqrt{M}e^{i\theta/2}$. In the case $\gamma_0 \neq 0$, $\tau \neq 0$, $g = 0$, we keep the minus sign in front of the square roots in (30) so that our branch definition will satisfy the requirement that ϕ and ψ have mean 1 when $\widehat{\Omega}_1 e^{\sigma t}$ is small. For larger values of $\widehat{\Omega}_1 e^{\sigma t}$, such as in Figure 1, our branch definition is equivalent to using the principal branch with a minus sign in front for the larger circular lobe, and a plus sign in front for the smaller lobe. Note that this definition of branch gives continuous dependence of the solution $\Omega(\xi)$ on $\widehat{\Omega}_1 e^{\sigma t}$. We similarly define a branch for the term $v(\Omega) = (1 + \beta\Omega + u(\Omega))^{1/2}$. It was shown earlier that the trace of $1 + \beta\Omega + u(\Omega)$ does not pass through zero as $\widehat{\Omega}_1 e^{\sigma t}$ is increased; however, it can cross the negative real axis. In this case, the square root definition given above maintains continuity of the term $(1 + \beta\Omega + u(\Omega))^{1/2}$.

The location ξ_s of a singularity in the complex plane can be determined by integration of (36) with the requirement that $\Omega(\xi) \sim \widehat{\Omega}_1 e^{\xi}$ as $\xi \to \infty$. It is easy to show (see [2]) that

$$\xi_s = \lim_{L \to \infty} \int_0^L \left(\frac{1}{F(\Omega\prime)} - \frac{1}{\Omega'} \right) d\Omega' + \ln L - \ln \widehat{\Omega}_1. \tag{38}$$

This expression defines two singularity positions, one for each sign of (principal) branch of F. If the sum of the first two terms in (38) is denoted by p, then the trajectory of the singularity ξ_p is given by

$$\alpha = -i(p - \ln \widehat{\Omega}_1) - \sigma t \ . \tag{39}$$

Thus, for a given value of surface tension τ, the distance of the singularity from the real line and the time it takes to reach it depend logarithmically on the size of the perturbation, for small perturbations. Although it is easy to determine p numerically, we instead solve (36) for $\Omega(\xi)$ so that the sheet location may be obtained in addition to the time of singularity formation.

5 Numerical Solutions

5.1 Numerical Method

We have obtained numerical solutions to equation (36) in the case $\beta = A = 0$ for various values of the surface tension coefficient τ. These solutions can be used to obtain the shape of the interface, location of singularity formation, and various other quantities of interest. In the computations, we set $\gamma_0 = 1$ and $\rho_1 + \rho_2 = 1$; together with a relevant length scale, such as the length of the computational box $L = 2\pi$, this choice makes our variables effectively dimensionless.

Input data for the computed solution consists of the surface tension coefficient τ and either the Fourier coefficient $\widehat{\Omega}_1 = \lim_{\xi \to -\infty} \dfrac{\Omega - \Gamma_0}{e^{\xi}}$ or the initial value $\Omega(\xi_0)$ of the function at the point $\xi_0 = i\alpha_0$. Equation (36) is then used to determine the higher Fourier coefficients $\widehat{\Omega}_2, \widehat{\Omega}_3, \ldots$ in the former case or integrated with respect to α to obtain the full solution in the latter case. Since Ω is of traveling wave form, its values at some later time t can be found by multiplying each Fourier coefficient $\widehat{\Omega}_k$ with $e^{k\sigma t}$. In practice, however, this procedure causes the small round off errors which exist in the high Fourier modes to be greatly magnified. As a result, we follow the second approach above by giving the value of $\Omega(\alpha = 0)$ and integrating equation (36) with respect to α. The solution at a later time t is then found by changing $\Omega(0)$ to a different value $\Omega_1(0)$ and again integrating. The time t of the solution passing through $\alpha = 0$, $\Omega_1(0)$ is given by

$$t = \frac{1}{\sigma} \ln \frac{\widetilde{\Omega}_1}{\widetilde{\Omega}},$$

where $\widetilde{\Omega}_1$, $\widetilde{\Omega}$ are the Fourier coefficients of $e^{i\alpha}$ in the solutions with initial values $\Omega_1(0)$, $\Omega(0)$, respectively.

Equation (36) was integrated with respect to α using the SLATEC differential equation solver DDEABM. The routine employs an Adams-Moulton predictor-corrector method with adaptive step size. The step-size is chosen so that the single step error does not exceed a user specified error per step. In each plot which is presented, a sufficient number of points are used to resolve all features.

Once the function $\Omega(\xi)$ has been determined, the other two variables in the localized approximation equations, ϕ and ψ, can be determined from (27). Integrating the relation $z_\alpha = \phi + \psi^* - 1$ then gives the position of the interface. The x constant of integration is determined by the requirement that $x(0) = 0$. It is also required that the physical interface $y(x)$ has no mean height; this fixes the y constant of integration. The unnormalized vortex sheet strength is determined from $\gamma = \gamma_0 + 2Re\ \omega$ with $\omega = \sigma\Phi_0\Psi_0(\Omega - \Gamma_0)$.

5.2 Computational Results

The solution $\Omega(\xi)$ to (36) is plotted in Figure 2 at four different times, for $\tau = .10$ The curves shown are obtained by plotting $Re\ (\Omega - \Gamma_0)$ on the x-axis and $Im\ (\Omega - \Gamma_0)$ on the y-axis, for α in the interval $[0, 2\pi]$. The corresponding interface positions and vortex sheet strengths are plotted in Figures 3 and 4. For this value of τ the linearized theory gives a cut-off wave number $k_c = 2.45$. Consequently, in the discrete system there are two unstable modes corresponding to $k = 1$ and $k = 2$, with the remaining modes neutrally stable. This case is intermediate between two extremes, namely that for which $k_c \to \infty$ as $\tau \to 0$ and that for which $k_c = 1$ and $\tau = 1/4$. These two extremes will be discussed subsequently.

In Figure 2a, the initial value of Ω is chosen as $\Omega(0) - \Gamma_0 = -.001$. By choosing the value of $\Omega - \Gamma_0$ to be a negative real number when $\alpha = 0$, we obtain an interfacial curve which satisfies $z(\alpha = 0) = 0$ and which has odd symmetry with respect to $\alpha = 0$. The value of $\Omega(0) - \Gamma_0$ in Figure 2a is small enough so that the motion is in the linear regime. Consequently, the solution is nearly circular and is closely approximated by $\Omega(\xi) = \Omega(0)e^{\xi}$, the solution to the linearized version of (36). As $\Omega(0)$ decreases, i.e., $|\Omega(0)|$ increases (this is equivalent to letting the solution $\Omega(\xi)$ with initial data given in Figure 2a evolve forward in time) the motion enters the non-linear regime and the curve $(Re\ \Omega(\alpha),\ Im\ \Omega(\alpha))$ deforms from its previously circular form. In Figure 2b, we see that a dimple forms on the curve as it tends to avoid the point $1 + \Omega^2 = 0$, which corresponds to a zero of the right hand side of equation (36). The other zeros of $F(\Omega)$ are farther away and do not visibly affect the shape of the curve. This zero and its consequent dimple lead to the formation of two relative vorticity maxima (see Figure 4b). This contrasts the zero surface tension case, in which a single maxima forms at $\alpha = 0$. Physically, surface tension produces a restoring force which prevents accumulation of vorticity at the center of the interface, where the curvature gradients are large. Instead, two relative vorticity maxima form on either side of the centerpoint, where the curvature gradient is smaller [24].

As $\Omega(0)$ is further decreased, the curve $(Re\ \Omega(\alpha),\ Im\ \Omega(\alpha))$ eventually passes through the point where $1+\Omega^2 = 0$ (Figure 2c). At this point $\Omega_\alpha = 0$ and the curve $(Re\ \Omega(\alpha), Im\ \Omega(\alpha))$ achieves a zero angled cusp. However, as demonstrated earlier, this point does not correspond to a singularity in any of the functions $\Omega(\xi)$, $\phi(\xi)$ or $\psi(\xi)$, and the evolution may be smoothly continued beyond it. The profile of the interface at this time indicates that the wave has just begun overturning.

Further decreasing $\Omega(0)$ produces a curve $(Re\ \Omega(\alpha),\ Im\ \Omega(\alpha))$ like that shown in Figure 2d. At this time the interface, shown in Figure 3d, has already developed the appearence of corners at the turning points in the $\mathcal{S}$-shaped curve. The sheet strength (Figure 4d) and curvature (Figure 5d) peak sharply at values of α corresponding to the location of the 'corners'. Between these peaks, the sheet strength is negative. Shortly after this time, the two intersecting lobes of the curve $(Re\ \Omega(\alpha), Im\ \Omega(\alpha))$ stretch out to infinity on the right. At this point, singularities appear on the interface and the unnormalized vortex sheet strength and curvature both blow-up.

Plots of the curvature corresponding to the interfacial profiles of Figure 3 are presented in Figure 5. Note that at intermediate times in the motion, a localized 'sine' like variation in the curvature develops and grows. At late times, two extremely sharp peaks of curvature shoot up from the edges of this localized perturbation, in the same locations as the vorticity peaks. The initial evolution of these eventual maxima are more closely detailed in Figure 6. Interestingly, the original perturbation shrinks as the sharp peaks grow. For other values of τ the maxima are located in different positions (this is detailed below) and the perturbed region stretches or compresses accordingly; otherwise the curvature resembles that in Figure 5 and will not be shown.

Many aspects of the behavior remain the same when the surface tension is increased, subject to the constraint that there is at least one unstable mode. Figures 7 and 8 depict the traveling wave solution for $\tau = .219$ and $k_c = 1.1$, at stages of the evolution analogous to those shown in Figures 3 and 4. Despite the similarities with the previous figures, there are some important qualitative differences. As shown in Figure 8, the two vorticity maxima are further separated in α. The interfacial profile assumes more of an extended $\mathcal{S}$ shape, and the amplitude of the deformation at the critical time is significantly larger than for the previous value of τ.

When surface tension is decreased, the opposite trends are observed. For example, consider $\tau = .0079$, for which the cut-off wave number is $k_c = 31.6$. Figures 9 and 10 contain the pertinent plots for this value of τ. In this case, the two vorticity and curvature maxima are closer in α than in the $\tau = .10$ case. Furthermore, the interfacial profile assumes a 'triangular' shape, and the amplitude at the critical time is greatly decreased from the previous cases. These trends continue as τ is further reduced: the separation of the vorticity and curvature maxima further decreases (see Figure 11), the sheet profile approaches a limiting 'triangular' shape at the critical time, and the size of the deformation at this time decreases. Figure 11 further illustrates the dependence of singularity position on surface tension. When $\tau = 0$, the only traveling wave solution is the flat sheet solution (5). The form of the traveling wave solution at the critical time for decreasing values of τ suggests an approach (in the weak sense) to the flat sheet solution as $\tau \to 0$.

6 Discussion

An interesting feature of the traveling wave solutions is that, for each value of τ, singularities form before there is any sign of interface roll-up. This is inconsistent with the results of [24] and [14], which show rolled-up interfaces in the presence of surface tension.[†] There are several possible explanations for this discrepency:

1. Interactions between positive and negative wavenumber modes, absent in the approximate equations, are necessary to produce roll-up.

2. The difference in initial data, i.e., traveling wave initial data versus linearized eigenfunction initial data in [24] and $\tau = 0$ linearized eigen-

[†]Our results are consistent with those of Pullen [23], in which the interface develops irregularities prior to roll-up similar to those produced by a singularity; however, the results of Baker and Nachbin [1] strongly suggest that the irregularities in Pullin's computations are numerical in nature.

function initial data in [14], are enough to cause the discrepency. We note that the traveling wave initial data and the linearized eigenfunction initial data are identical to $O(\varepsilon^2)$, where ε is the height of the initial disturbance; the traveling wave and $\tau = 0$ linearized eigenfunction data differ by $O(\varepsilon)$.

3. Smoothing operations remove a singularity which occurs prior to roll-up and enable the computations to continue into the roll-up regime.

The careful computations of Hou et. al. [14] show roll-up occuring prior to the formation of singularities and therefore render 1 and 2 more plausible explanations of our lack of roll-up. The discrepency may be related to the late time vorticity distribution for the traveling wave solution, which contains two very narrow peaks with a region of oppositely signed vorticity between them. This distribution is consistent with an interfacial profile that stretches but does not roll-up. The vorticity exhibits this form regardless of the value of surface tension, as long as there is at least one unstable mode. However, this particular distribution may not be generic for other initial data.

For $\tau \gtrsim .04$, the computations in [24] also show a doubly peaked vorticity distribution, although γ remains positive for all α. In this situation, they find that the sheet does not roll up. On the other hand, for smaller τ, they find the vorticity accumulates in a single peak at the center of the curve, leading to roll up. It is possible that the method of [24] does not differentiate between two closely spaced vorticity peaks, since their method involves insertion and redistribution of vortex sheet markers, using linear interpolation to position and define the sheet strength of the redistributed markers. This is an effective smoothing operation.

In contrast, Beale and Lowengrub [5] find that the vorticity distribution always forms two peaks and that the separation of these peaks decreases with decreasing τ. Their results suggest that these two features are generic. They use the $\tau = 0$ linearized eigenfunctions for initial data, and consider values of surface tension for which their code runs a reasonable length of time. The presence of two vorticty peaks was taken as evidence of the existence of two singularities in the complex plane which approach each other as τ decreases. Although their vorticty distribution strongly resembles that for the traveling wave solution, their code stops before the vorticity between the peaks goes negative. The computations of Hou et. al. [14] extend the solution for this initial data to much longer times. However, their data does not show the vorticity going negative until just before a late time singularity, at which point the (rolled-up) interface and the vorticity distribution have little resemblance to the traveling wave solution. Nevertheless, the exponent of curvature blow-up is the same as that predicted by the traveling wave solution. (need to talk more with Mike or John about this).

7 Conclusion

The analysis and computations presented above show that, in the approximate theory, singularities develop on the fluid interface in finite time. The behavior of the sheet position z, the vorticity strength γ, and the curvature κ in the vicinity of a singularity ζ_s is given by $z_\xi \sim a_1(\xi - \xi_s)^{-\frac{2}{3}}$, $\gamma \sim a_2(\xi - \xi_s)^{-\frac{2}{3}}$, and $\kappa \sim a_2(\xi - \xi_s)^{-\frac{1}{3}}$, where a_1–a_3 are constants. The theory is applied outside of its range of formal validity, so the finite time occurence of these singularities in the full equations must be accepted as merely plausible. Numerical computations using the traveling wave initial conditions are being performed by G. Baker and A. Nachbin in order to verify the asymptotic predictions.

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List of Figure Captions

1. The form of the curve $\mathcal{C}$, after it has passed through the origin.

2. Phase plots of the solution $\Omega(\alpha) - \Gamma_0$ to (36), for $\alpha \in [0, 2\pi]$. Surface tension is set at $\tau = .10$. The plots (a)–(d) correspond to initial values $\Omega(0) - \Gamma_0$ = -.001, -.15, -.34, -.569 and times t = 0., 11.02, 12.25, and 13.15. The marker ($\times$) denotes the position where $1 + \Omega^2 = 0$.

3. The interface profiles $(x(\alpha), y(\alpha))$ corresponding to the solutions in Figure 2. Note that the $t = 0$ curve is plotted on a different scale.

4. The vortex sheet strength γ versus α, corresponding to the solutions in Figure 2. The plots are on different scales. In (d), the sheet strength peaks at $\gamma(\pm.87) = 173$ (not shown); the strength at $\alpha = 0$ is $\gamma(0) = -.14$.

5. The curvature κ versus α, corresponding to the solutions in Figure 2. The plots are on different scales. In (d), the curvature peaks at $\kappa(\pm.87) = \pm 26.5$ (not shown).

6. The initial evolution of curvature spikes. The plots correspond to initial values $\Omega(0) - \Gamma_0$ = -.54, -.55, -.56 and times $t = 13.05, 13.08, 13.12$.

7. The interface profiles $(x(\alpha), y(\alpha))$ for $\tau = .219$. The curves (a)–(d) correspond to initial values $\Omega(0) - \Gamma_0$ = -.001, -.4, -.47, -.5402, and times t = 0., 35.7, 37.1, 38.5. The $t = 0$ curve is plotted on a different scale.

8. The vortex sheet strength γ versus α, corresponding to the profiles in Figure 7. The plots are on different scales. In (d), the sheet strength peaks at $\gamma(\pm 2.31) = 303$ (not shown); the strength at $\alpha = 0$ is $\gamma(0) = -.08$.

9. The interface profiles $(x(\alpha), y(\alpha))$ for $\tau = .0079$. The curves (a)–(d) correspond to initial values $\Omega(0) - \Gamma_0$ = -.00001, -.001, -.175, -.5718 and times t = 0., 6.7, 8.1, 8.2. The $t = 0$ curve is plotted on a different scale.

10. The vortex sheet strength γ versus α, corresponding to the profiles in Figure 9. The plots are on different scales. In (d), the sheet strength peaks at $\gamma(\pm.06) = 145$ (not shown); the strength at $\alpha = 0$ is $\gamma(0) = -.14$.

11. The distance from the origin of the singularity on the interface, measured in terms of the arclength s along the interface, and plotted as a function of τ. For the range of τ shown, the dependence is approximately linear.