

Sample Paper for the aomart Class

By AMERICAN MATHEMATICAL SOCIETY and BORIS VEYTSMAN,
WITH APPENDIX BY FRODO BAGGINS and BILBO BAGGINS and WITH AFTERWORD
BY BILBO BAGGINS

Abstract

This is a test file for `aomart` class based on the `testmath.tex` file from the `amsmath` distribution.

It was changed to test the features of the Annals of Mathematics class.

Contents

19	1. Introduction	18
20	2. Enumeration of Hamiltonian paths in a graph	18
21	3. Main theorem	19
22	4. Application	22
23	5. Secret key exchanges	23
24	6. Review	23
25	7. One-way complexity	28
26	8. Various font features of the <code>amsmath</code> package	35
27	8.1. Bold versions of special symbols	35
28	8.2. “Poor man’s bold”	35
29	9. Compound symbols and other features	36
30	9.1. Multiple integral signs	36
31	9.2. Over and under arrows	36
32	9.3. Dots	36
33	9.4. Accents in math	37
34	9.5. Dot accents	37
35	9.6. Roots	37

Keywords: Hamiltonian paths, Typesetting

AMS Classification: Primary: 1AB5 (matsc2020), 2FD5 (matsc2020); Secondary: FFFF (matsc2020), G25 (matsc2020).

The class was commissioned by Annals of Mathematics.

© 2008–2020 Boris Veytsman.

<u>1</u>	9.7. Boxed formulas	37
<u>2</u>	9.8. Extensible arrows	38
<u>3</u>	9.9. <code>\overset</code> , <code>\underset</code> , and <code>\sideset</code>	38
<u>4</u>	9.10. The <code>\text</code> command	38
<u>5</u>	9.11. Operator names	38
<u>6</u>	9.12. <code>\mod</code> and its relatives	39
<u>7</u>	9.13. Fractions and related constructions	39
<u>8</u>	9.14. Continued fractions	41
<u>9</u>	9.15. Smash	41
<u>10</u>	9.16. The ‘cases’ environment	41
<u>11</u>	9.17. Matrix	42
<u>12</u>	9.18. The <code>\substack</code> command	43
<u>13</u>	9.19. Big-g-g delimiters	44
<u>14</u>	9.20. Acknowledgements	44
<u>15</u>	References	44

161718

1. Introduction

192021222324252627

2. Enumeration of Hamiltonian paths in a graph

282930313233343536373839404142

This paper demonstrates the use of `aomart` class. It is based on `testmath.tex` from $\mathcal{A}\mathcal{M}\mathcal{S}\text{-}\mathcal{L}\mathcal{A}\mathcal{T}\mathcal{E}\mathcal{X}$ distribution. The text is (slightly) reformatted according to the requirements of the `aomart` style. See also [12, 22, 17, 1, 16, 15, 24, 23, 6].

It is always a pleasure to cite Knuth [9].

Let $\mathbf{A} = (a_{ij})$ be the adjacency matrix of graph G . The corresponding Kirchhoff matrix $\mathbf{K} = (k_{ij})$ is obtained from \mathbf{A} by replacing in $-\mathbf{A}$ each diagonal entry by the degree of its corresponding vertex; i.e., the i th diagonal entry is identified with the degree of the i th vertex. It is well known that

$$(1) \quad \det \mathbf{K}(i|i) = \text{the number of spanning trees of } G, \quad i = 1, \dots, n$$

where $\mathbf{K}(i|i)$ is the i th principal submatrix of \mathbf{K} .

`\det\mathbf{K}(i|i)=\text{the number of spanning trees of }G`,

Let $C_{i(j)}$ be the set of graphs obtained from G by attaching edge $(v_i v_j)$ to each spanning tree of G . Denote by $C_i = \bigcup_j C_{i(j)}$. It is obvious that the collection of Hamiltonian cycles is a subset of C_i . Note that the cardinality of C_i is $k_{ii} \det \mathbf{K}(i|i)$. Let $\widehat{X} = \{\hat{x}_1, \dots, \hat{x}_n\}$.

`\wh X=\{\hat{x}_1,\dots,\hat{x}_n\}`

Are these
quotations
necessary?

1 Define multiplication for the elements of \widehat{X} by

$$\hat{x}_i \hat{x}_j = \hat{x}_j \hat{x}_i, \quad \hat{x}_i^2 = 0, \quad i, j = 1, \dots, n.$$

2 3 4 Let $\hat{k}_{ij} = k_{ij} \hat{x}_j$ and $\hat{k}_{ij} = -\sum_{j \neq i} \hat{k}_{ij}$. Then the number of Hamiltonian cycles H_c is given by the relation [13]

$$\hat{k}_{ij} = -\sum_{j \neq i} \hat{k}_{ij}$$

$$\text{(3)} \quad \left(\prod_{j=1}^n \hat{x}_j \right) H_c = \frac{1}{2} \hat{k}_{ij} \det \widehat{\mathbf{K}}(i|i), \quad i = 1, \dots, n.$$

5 6 7 8 The task here is to express (3) in a form free of any $\hat{x}_i, i = 1, \dots, n$. The result also leads to the resolution of enumeration of Hamiltonian paths in a graph.

9 It is well known that the enumeration of Hamiltonian cycles and paths 10 in a complete graph K_n and in a complete bipartite graph $K_{n_1 n_2}$ can only 11 be found from *first combinatorial principles* [7]. One wonders if there exists a 12 formula which can be used very efficiently to produce K_n and $K_{n_1 n_2}$. Recently, 13 using Lagrangian methods, Goulden and Jackson have shown that H_c can be 14 expressed in terms of the determinant and permanent of the adjacency matrix 15 [5]. However, the formula of Goulden and Jackson determines neither K_n nor 16 $K_{n_1 n_2}$ effectively. In this paper, using an algebraic method, we parametrize 17 the adjacency matrix. The resulting formula also involves the determinant 18 and permanent, but it can easily be applied to K_n and $K_{n_1 n_2}$. In addition, 19 we eliminate the permanent from H_c and show that H_c can be represented by 20 a determinantal function of multivariables, each variable with domain $\{0, 1\}$. 21 Furthermore, we show that H_c can be written by number of spanning trees of 22 subgraphs. Finally, we apply the formulas to a complete multigraph $K_{n_1 \dots n_p}$. 23

24 The conditions $a_{ij} = a_{ji}, i, j = 1, \dots, n$, are not required in this paper. 25 All formulas can be extended to a digraph simply by multiplying H_c by 2. 26 Some other discussion can be found in [4, 3]. 27

28 29 3. Main theorem

30 *Notation.* For $p, q \in P$ and $n \in \omega$ we write $(q, n) \leq (p, n)$ if $q \leq p$ and 31 $A_{q,n} = A_{p,n}$.

32 $\backslash\text{begin}\{\text{notation}\}$ For $p, q \in P$ and $n \in \omega$

33 \dots

34 $\backslash\text{end}\{\text{notation}\}$

35 Let $\mathbf{B} = (b_{ij})$ be an $n \times n$ matrix. Let $\mathbf{n} = \{1, \dots, n\}$. Using the properties 36 of (2), it is readily seen that 37

38 LEMMA 3.1.

$$\text{(4)} \quad \prod_{i \in \mathbf{n}} \left(\sum_{j \in \mathbf{n}} b_{ij} \hat{x}_i \right) = \left(\prod_{i \in \mathbf{n}} \hat{x}_i \right) \text{per } \mathbf{B}$$

1 where $\text{per } \mathbf{B}$ is the permanent of \mathbf{B} .

2 Let $\widehat{Y} = \{\hat{y}_1, \dots, \hat{y}_n\}$. Define multiplication for the elements of \widehat{Y} by

3
4 (5)
$$\hat{y}_i \hat{y}_j + \hat{y}_j \hat{y}_i = 0, \quad i, j = 1, \dots, n.$$

5 Then, it follows that

6 LEMMA 3.2.
7

8 (6)
$$\prod_{i \in \mathbf{n}} \left(\sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j \right) = \left(\prod_{i \in \mathbf{n}} \hat{y}_i \right) \det \mathbf{B}.$$

9
10 Note that all basic properties of determinants are direct consequences of
11 Lemma 3.2. Write

12 (7)
$$\sum_{j \in \mathbf{n}} b_{ij} \hat{y}_j = \sum_{j \in \mathbf{n}} b_{ij}^{(\lambda)} \hat{y}_j + (b_{ii} - \lambda_i) \hat{y}_i$$

13 where

14 (8)
$$b_{ii}^{(\lambda)} = \lambda_i, \quad b_{ij}^{(\lambda)} = b_{ij}, \quad i \neq j.$$

15 Let $\mathbf{B}^{(\lambda)} = (b_{ij}^{(\lambda)})$. By (6) and (7), it is straightforward to show the following
16 result:

17 THEOREM 3.3.
18

19 (9)
$$\det \mathbf{B} = \sum_{l=0}^n \sum_{I_l \subseteq \mathbf{n}} \prod_{i \in I_l} (b_{ii} - \lambda_i) \det \mathbf{B}^{(\lambda)}(I_l | I_l),$$

20 where $I_l = \{i_1, \dots, i_l\}$ and $\mathbf{B}^{(\lambda)}(I_l | I_l)$ is the principal submatrix (obtained from
21 $\mathbf{B}^{(\lambda)}$ by deleting its i_1, \dots, i_l rows and columns).

22 Remark 3.1 (convention). Let \mathbf{M} be an $n \times n$ matrix. The convention
23 $\mathbf{M}(\mathbf{n} | \mathbf{n}) = 1$ has been used in (9) and hereafter.

24 Before proceeding with our discussion, we pause to note that Theorem 3.3
25 yields immediately a fundamental formula which can be used to compute the
26 coefficients of a characteristic polynomial [14]:

27 COROLLARY 3.4. Write $\det(\mathbf{B} - x\mathbf{I}) = \sum_{l=0}^n (-1)^l b_l x^l$. Then

28 (10)
$$b_l = \sum_{I_l \subseteq \mathbf{n}} \det \mathbf{B}(I_l | I_l).$$

29 Let

30 (11)
$$\mathbf{K}(t, t_1, \dots, t_n) = \begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \dots & \dots & \dots & \dots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix},$$

31
32
33
34
35
36
37
38
39
40
41
42

$$\begin{pmatrix} D_1 t & -a_{12} t_2 & \dots & -a_{1n} t_n \\ -a_{21} t_1 & D_2 t & \dots & -a_{2n} t_n \\ \vdots & \vdots & \ddots & \vdots \\ -a_{n1} t_1 & -a_{n2} t_2 & \dots & D_n t \end{pmatrix}$$

where

$$(12) \quad D_i = \sum_{j \in \mathbf{n}} a_{ij} t_j, \quad i = 1, \dots, n.$$

Set

$$D(t_1, \dots, t_n) = \frac{\delta}{\delta t} \det \mathbf{K}(t, t_1, \dots, t_n)|_{t=1}.$$

Then

$$(13) \quad D(t_1, \dots, t_n) = \sum_{i \in \mathbf{n}} D_i \det \mathbf{K}(t = 1, t_1, \dots, t_n; i|i),$$

where $\mathbf{K}(t = 1, t_1, \dots, t_n; i|i)$ is the i th principal submatrix of $\mathbf{K}(t = 1, t_1, \dots, t_n)$.

Theorem 3.3 leads to

$$(14) \quad \det \mathbf{K}(t_1, t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|} t^{n-|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}).$$

Note that

$$(15) \quad \det \mathbf{K}(t = 1, t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}) = 0.$$

Let $t_i = \hat{x}_i, i = 1, \dots, n$. Lemma 3.1 yields

$$(16) \quad \left(\sum_{i \in \mathbf{n}} a_i x_i \right) \det \mathbf{K}(t = 1, x_1, \dots, x_n; l|l) \\ = \left(\prod_{i \in \mathbf{n}} \hat{x}_i \right) \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\}).$$

$$\begin{aligned} & \biggl(\sum_{i \in \mathbf{n}} a_i x_i \biggr) \det \mathbf{K}(t=1, x_1, \dots, x_n; l | l) \\ & = \biggl(\prod_{i \in \mathbf{n}} \hat{x}_i \biggr) \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\}). \end{aligned}$$

By (3), (6), and (7), we have

PROPOSITION 3.5.

$$(17) \quad H_c = \frac{1}{2n} \sum_{l=0}^n (-1)^l D_l,$$

where

$$(18) \quad D_l = \sum_{I_l \subseteq \mathbf{n}} D(t_1, \dots, t_n) \Big|_{t_i = \begin{cases} 0, & \text{if } i \in I_l \\ 1, & \text{otherwise} \end{cases}, \quad i=1, \dots, n.$$

4. Application

We consider here the applications of Theorems 5.1 and 5.2 on page 23 to a complete multipartite graph $K_{n_1 \dots n_p}$. It can be shown that the number of spanning trees of $K_{n_1 \dots n_p}$ may be written

$$(19) \quad T = n^{p-2} \prod_{i=1}^p (n - n_i)^{n_i-1}$$

where

$$(20) \quad n = n_1 + \dots + n_p.$$

It follows from Theorems 5.1 and 5.2 that

$$(21) \quad H_c = \frac{1}{2n} \sum_{l=0}^n (-1)^l (n-l)^{p-2} \sum_{l_1 + \dots + l_p = l} \prod_{i=1}^p \binom{n_i}{l_i} \cdot [(n-l) - (n_i - l_i)]^{n_i - l_i} \cdot \left[(n-l)^2 - \sum_{j=1}^p (n_i - l_i)^2 \right].$$

...

and

$$(22) \quad H_c = \frac{1}{2} \sum_{l=0}^{n-1} (-1)^l (n-l)^{p-2} \sum_{l_1 + \dots + l_p = l} \prod_{i=1}^p \binom{n_i}{l_i} \cdot [(n-l) - (n_i - l_i)]^{n_i - l_i} \left(1 - \frac{l_p}{n_p} \right) [(n-l) - (n_p - l_p)].$$

The enumeration of H_c in a $K_{n_1 \dots n_p}$ graph can also be carried out by Theorem 7.2 or 7.3 together with the algebraic method of (2). Some elegant representations may be obtained. For example, H_c in a $K_{n_1 n_2 n_3}$ graph may be written

$$(23) \quad H_c = \frac{n_1! n_2! n_3!}{n_1 + n_2 + n_3} \sum_i \left[\binom{n_1}{i} \binom{n_2}{n_3 - n_1 + i} \binom{n_3}{n_3 - n_2 + i} + \binom{n_1 - 1}{i} \binom{n_2 - 1}{n_3 - n_1 + i} \binom{n_3 - 1}{n_3 - n_2 + i} \right].$$

5. Secret key exchanges

Modern cryptography is fundamentally concerned with the problem of secure private communication. A Secret Key Exchange is a protocol where Alice and Bob, having no secret information in common to start, are able to agree on a common secret key, conversing over a public channel. The notion of a Secret Key Exchange protocol was first introduced in the seminal paper of Diffie and Hellman [2]. [2] presented a concrete implementation of a Secret Key Exchange protocol, dependent on a specific assumption (a variant on the discrete log), specially tailored to yield Secret Key Exchange. Secret Key Exchange is of course trivial if trapdoor permutations exist. However, there is no known implementation based on a weaker general assumption.

The concept of an informationally one-way function was introduced in [8]. We give only an informal definition here:

Definition 5.1 (one way). A polynomial time computable function $f = \{f_k\}$ is informationally one-way if there is no probabilistic polynomial time algorithm which (with probability of the form $1 - k^{-e}$ for some $e > 0$) returns on input $y \in \{0, 1\}^k$ a random element of $f^{-1}(y)$.

In the non-uniform setting [8] show that these are not weaker than one-way functions:

THEOREM 5.1 ([8] (non-uniform)). *The existence of informationally one-way functions implies the existence of one-way functions.*

We will stick to the convention introduced above of saying “non-uniform” before the theorem statement when the theorem makes use of non-uniformity. It should be understood that if nothing is said then the result holds for both the uniform and the non-uniform models.

It now follows from Theorem 5.1 that

THEOREM 5.2 (non-uniform). *Weak SKE implies the existence of a one-way function.*

More recently, the polynomial-time, interior point algorithms for linear programming have been extended to the case of convex quadratic programs [19, 21], certain linear complementarity problems [11, 18], and the nonlinear complementarity problem [10]. The connection between these algorithms and the classical Newton method for nonlinear equations is well explained in [11].

6. Review

We begin our discussion with the following definition:

1 *Definition 6.1.* A function $H: \mathfrak{R}^n \rightarrow \mathfrak{R}^n$ is said to be *B-differentiable*
2 at the point z if (i) H is Lipschitz continuous in a neighborhood of z , and
3 (ii) there exists a positive homogeneous function $BH(z): \mathfrak{R}^n \rightarrow \mathfrak{R}^n$, called the
4 *B-derivative* of H at z , such that

$$\lim_{v \rightarrow 0} \frac{H(z+v) - H(z) - BH(z)v}{\|v\|} = 0.$$

7 The function H is *B-differentiable in set S* if it is B-differentiable at every
8 point in S . The B-derivative $BH(z)$ is said to be *strong* if

$$\lim_{(v,v') \rightarrow (0,0)} \frac{H(z+v) - H(z+v') - BH(z)(v-v')}{\|v-v'\|} = 0.$$

12 LEMMA 6.1. *There exists a smooth function $\psi_0(z)$ defined for $|z| > 1 - 2a$
13 satisfying the following properties:*

- 14 (i) $\psi_0(z)$ is bounded above and below by positive constants $c_1 \leq \psi_0(z) \leq c_2$.
- 15 (ii) If $|z| > 1$, then $\psi_0(z) = 1$.
- 16 (iii) For all z in the domain of ψ_0 , $\Delta_0 \ln \psi_0 \geq 0$.
- 17 (iv) If $1 - 2a < |z| < 1 - a$, then $\Delta_0 \ln \psi_0 \geq c_3 > 0$.

18 *Proof.* We choose $\psi_0(z)$ to be a radial function depending only on $r = |z|$.
19 Let $h(r) \geq 0$ be a suitable smooth function satisfying $h(r) \geq c_3$ for $1 - 2a <$
20 $|z| < 1 - a$, and $h(r) = 0$ for $|z| > 1 - \frac{a}{2}$. The radial Laplacian
21

$$\Delta_0 \ln \psi_0(r) = \left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \ln \psi_0(r)$$

24 has smooth coefficients for $r > 1 - 2a$. Therefore, we may apply the existence
25 and uniqueness theory for ordinary differential equations. Simply let $\ln \psi_0(r)$
26 be the solution of the differential equation
27

$$\left(\frac{d^2}{dr^2} + \frac{1}{r} \frac{d}{dr} \right) \ln \psi_0(r) = h(r)$$

30 with initial conditions given by $\ln \psi_0(1) = 0$ and $\ln \psi_0'(1) = 0$.

31 Next, let D_ν be a finite collection of pairwise disjoint disks, all of which
32 are contained in the unit disk centered at the origin in C . We assume that
33 $D_\nu = \{z \mid |z - z_\nu| < \delta\}$. Suppose that $D_\nu(a)$ denotes the smaller concentric
34 disk $D_\nu(a) = \{z \mid |z - z_\nu| \leq (1 - 2a)\delta\}$. We define a smooth weight function
35 $\Phi_0(z)$ for $z \in C - \bigcup_\nu D_\nu(a)$ by setting $\Phi_0(z) = 1$ when $z \notin \bigcup_\nu D_\nu$ and
36 $\Phi_0(z) = \psi_0((z - z_\nu)/\delta)$ when z is an element of D_ν . It follows from Lemma 6.1
37 that Φ_0 satisfies the properties:

- 38 (i) $\Phi_0(z)$ is bounded above and below by positive constants $c_1 \leq \Phi_0(z) \leq$
39 c_2 .
- 40 (ii) $\Delta_0 \ln \Phi_0 \geq 0$ for all $z \in C - \bigcup_\nu D_\nu(a)$, the domain where the function
41 Φ_0 is defined.

42

1 (iii) $\Delta_0 \ln \Phi_0 \geq c_3 \delta^{-2}$ when $(1 - 2a)\delta < |z - z_\nu| < (1 - a)\delta$.
2 Let A_ν denote the annulus $A_\nu = \{(1 - 2a)\delta < |z - z_\nu| < (1 - a)\delta\}$, and
3 set $A = \bigcup_\nu A_\nu$. The properties (2) and (3) of Φ_0 may be summarized as
4 $\Delta_0 \ln \Phi_0 \geq c_3 \delta^{-2} \chi_A$, where χ_A is the characteristic function of A . \square

5 Suppose that α is a nonnegative real constant. We apply Proposition 3.5
6 with $\Phi(z) = \Phi_0(z)e^{\alpha|z|^2}$. If $u \in C_0^\infty(R^2 - \bigcup_\nu D_\nu(a))$, assume that \mathcal{D} is a
7 bounded domain containing the support of u and $A \subset \mathcal{D} \subset R^2 - \bigcup_\nu D_\nu(a)$. A
8 calculation gives

$$\int_{\mathcal{D}} |\bar{\partial}u|^2 \Phi_0(z)e^{\alpha|z|^2} \geq c_4 \alpha \int_{\mathcal{D}} |u|^2 \Phi_0 e^{\alpha|z|^2} + c_5 \delta^{-2} \int_A |u|^2 \Phi_0 e^{\alpha|z|^2}.$$

12 The boundedness, property (1) of Φ_0 , then yields

$$\int_{\mathcal{D}} |\bar{\partial}u|^2 e^{\alpha|z|^2} \geq c_6 \alpha \int_{\mathcal{D}} |u|^2 e^{\alpha|z|^2} + c_7 \delta^{-2} \int_A |u|^2 e^{\alpha|z|^2}.$$

16 Let $B(X)$ be the set of blocks of Λ_X and let $b(X) = |B(X)|$. If $\phi \in Q_X$
17 then ϕ is constant on the blocks of Λ_X .

$$(24) \quad P_X = \{\phi \in M \mid \Lambda_\phi = \Lambda_X\}, \quad Q_X = \{\phi \in M \mid \Lambda_\phi \geq \Lambda_X\}.$$

20 If $\Lambda_\phi \geq \Lambda_X$ then $\Lambda_\phi = \Lambda_Y$ for some $Y \geq X$ so that

$$Q_X = \bigcup_{Y \geq X} P_Y.$$

24 Thus by Möbius inversion

$$|P_Y| = \sum_{X \geq Y} \mu(Y, X) |Q_X|.$$

28 Thus there is a bijection from Q_X to $W^{B(X)}$. In particular $|Q_X| = w^{b(X)}$.

29 Next note that $b(X) = \dim X$. We see this by choosing a basis for X
30 consisting of vectors v^k defined by

$$v_i^k = \begin{cases} 1 & \text{if } i \in \Lambda_k, \\ 0 & \text{otherwise.} \end{cases}$$

34 $\backslash [v^{\wedge k}]_i =$
35 $\backslash \begin{cases} 1 & \& \text{if } i \in \Lambda_k, \\ 0 & \& \text{otherwise.} \end{cases} \backslash \backslash$
36 $\backslash \backslash$
37 $\backslash]$

38 LEMMA 6.2. *Let \mathcal{A} be an arrangement. Then*

$$\chi(\mathcal{A}, t) = \sum_{\mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})}.$$

42

1 In order to compute R'' recall the definition of $S(X, Y)$ from Lemma 3.1.
2 Since $H \in \mathcal{B}$, $\mathcal{A}_H \subseteq \mathcal{B}$. Thus if $T(\mathcal{B}) = Y$ then $\mathcal{B} \in S(H, Y)$. Let $L'' = L(\mathcal{A}'')$.
3 Then

$$\begin{aligned}
 \frac{4}{R''} &= \sum_{H \in \mathcal{B} \subseteq \mathcal{A}} (-1)^{|\mathcal{B}|} t^{\dim T(\mathcal{B})} \\
 \frac{5}{} &= \sum_{Y \in L''} \sum_{\mathcal{B} \in S(H, Y)} (-1)^{|\mathcal{B}|} t^{\dim Y} \\
 \frac{6}{} &= - \sum_{Y \in L''} \sum_{\mathcal{B} \in S(H, Y)} (-1)^{|\mathcal{B} - \mathcal{A}_H|} t^{\dim Y} \\
 \frac{7}{(25)} &= - \sum_{Y \in L''} \mu(H, Y) t^{\dim Y} \\
 \frac{8}{} &= -\chi(\mathcal{A}'', t).
 \end{aligned}$$

14 COROLLARY 6.3. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple of arrangements. Then

$$\pi(\mathcal{A}, t) = \pi(\mathcal{A}', t) + t\pi(\mathcal{A}'', t).$$

17 Definition 6.2. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple with respect to the hyperplane
18 $H \in \mathcal{A}$. Call H a separator if $T(\mathcal{A}) \notin L(\mathcal{A}')$.

20 COROLLARY 6.4. Let $(\mathcal{A}, \mathcal{A}', \mathcal{A}'')$ be a triple with respect to $H \in \mathcal{A}$.

21 (i) If H is a separator then

$$\mu(\mathcal{A}) = -\mu(\mathcal{A}'')$$

24 and hence

$$|\mu(\mathcal{A})| = |\mu(\mathcal{A}'')|.$$

26 (ii) If H is not a separator then

$$\mu(\mathcal{A}) = \mu(\mathcal{A}') - \mu(\mathcal{A}'')$$

29 and

$$|\mu(\mathcal{A})| = |\mu(\mathcal{A}')| + |\mu(\mathcal{A}'')|.$$

32 Proof. It follows from Theorem 5.1 that $\pi(\mathcal{A}, t)$ has leading term

$$(-1)^{r(\mathcal{A})} \mu(\mathcal{A}) t^{r(\mathcal{A})}.$$

35 The conclusion follows by comparing coefficients of the leading terms on both
36 sides of the equation in Corollary 6.3. If H is a separator then $r(\mathcal{A}') < r(\mathcal{A})$
37 and there is no contribution from $\pi(\mathcal{A}', t)$. \square

38 The Poincaré polynomial of an arrangement will appear repeatedly in
39 these notes. It will be shown to equal the Poincaré polynomial of the graded
40 algebras which we are going to associate with \mathcal{A} . It is also the Poincaré poly-
41 nomial of the complement $M(\mathcal{A})$ for a complex arrangement. Here we prove
42

1
2
3
4
5
6
7
8
9
10
11
12
13
14
15
16
17
18
19
20
21
22
23
24
25
26
27
28
29
30
31
32
33
34
35
36
37
38
39
40
41
42

Figure 1. $Q(\mathcal{A}_1) = xyz(x - z)(x + z)(y - z)(y + z)$

Figure 2. $Q(\mathcal{A}_2) = xyz(x + y + z)(x + y - z)(x - y + z)(x - y - z)$

that the Poincaré polynomial is the chamber counting function for a real arrangement. The complement $M(\mathcal{A})$ is a disjoint union of chambers

$$M(\mathcal{A}) = \bigcup_{C \in \text{Cham}(\mathcal{A})} C.$$

The number of chambers is determined by the Poincaré polynomial as follows.

THEOREM 6.5. *Let $\mathcal{A}_{\mathbf{R}}$ be a real arrangement. Then*

$$|\text{Cham}(\mathcal{A}_{\mathbf{R}})| = \pi(\mathcal{A}_{\mathbf{R}}, 1).$$

Proof. We check the properties required in Corollary 6.4: (i) follows from $\pi(\Phi_l, t) = 1$, and (ii) is a consequence of Corollary 3.4. \square

THEOREM 6.6. *Let ϕ be a protocol for a random pair (X, Y) . If one of $\sigma_\phi(x', y)$ and $\sigma_\phi(x, y')$ is a prefix of the other and $(x, y) \in S_{X, Y}$, then*

$$\langle \sigma_j(x', y) \rangle_{j=1}^\infty = \langle \sigma_j(x, y) \rangle_{j=1}^\infty = \langle \sigma_j(x, y') \rangle_{j=1}^\infty.$$

1 *Proof.* We show by induction on i that

$$\langle \sigma_j(x', y) \rangle_{j=1}^i = \langle \sigma_j(x, y) \rangle_{j=1}^i = \langle \sigma_j(x, y') \rangle_{j=1}^i.$$

4 The induction hypothesis holds vacuously for $i = 0$. Assume it holds for
5 $i - 1$, in particular $[\sigma_j(x', y)]_{j=1}^{i-1} = [\sigma_j(x, y')]_{j=1}^{i-1}$. Then one of $[\sigma_j(x', y)]_{j=i}^{\infty}$
6 and $[\sigma_j(x, y')]_{j=i}^{\infty}$ is a prefix of the other which implies that one of $\sigma_i(x', y)$
7 and $\sigma_i(x, y')$ is a prefix of the other. If the i th message is transmitted by
8 P_X then, by the separate-transmissions property and the induction hypothe-
9 sis, $\sigma_i(x, y) = \sigma_i(x, y')$, hence one of $\sigma_i(x, y)$ and $\sigma_i(x', y)$ is a prefix of the
10 other. By the implicit-termination property, neither $\sigma_i(x, y)$ nor $\sigma_i(x', y)$ can
11 be a proper prefix of the other, hence they must be the same and $\sigma_i(x', y) =$
12 $\sigma_i(x, y) = \sigma_i(x, y')$. If the i th message is transmitted by P_Y then, symmet-
13 rically, $\sigma_i(x, y) = \sigma_i(x', y)$ by the induction hypothesis and the separate-
14 transmissions property, and, then, $\sigma_i(x, y) = \sigma_i(x, y')$ by the implicit-termination
15 property, proving the induction step. \square

16 If ϕ is a protocol for (X, Y) , and $(x, y), (x', y)$ are distinct inputs in $S_{X,Y}$,
17 then, by the correct-decision property, $\langle \sigma_j(x, y) \rangle_{j=1}^{\infty} \neq \langle \sigma_j(x', y) \rangle_{j=1}^{\infty}$.

18 Equation (25) defined P_Y 's ambiguity set $S_{X|Y}(y)$ to be the set of possible
19 X values when $Y = y$. The last corollary implies that for all $y \in S_Y$, the
20 multiset¹ of codewords $\{\sigma_\phi(x, y) : x \in S_{X|Y}(y)\}$ is prefix free.

22

23 7. One-way complexity

24 $\hat{C}_1(X|Y)$, the one-way complexity of a random pair (X, Y) , is the number
25 of bits P_X must transmit in the worst case when P_Y is not permitted to transmit
26 any feedback messages. Starting with $S_{X,Y}$, the support set of (X, Y) , we define
27 $G(X|Y)$, the *characteristic hypergraph* of (X, Y) , and show that

$$\hat{C}_1(X|Y) = \lceil \log \chi(G(X|Y)) \rceil.$$

30 Let (X, Y) be a random pair. For each y in S_Y , the support set of Y ,
31 equation (25) defined $S_{X|Y}(y)$ to be the set of possible x values when $Y = y$.
32 The *characteristic hypergraph* $G(X|Y)$ of (X, Y) has S_X as its vertex set and
33 the hyperedge $S_{X|Y}(y)$ for each $y \in S_Y$.

34 We can now prove a continuity theorem.

35 **THEOREM 7.1.** *Let $\Omega \subset \mathbf{R}^n$ be an open set, let $u \in BV(\Omega; \mathbf{R}^m)$, and let*

$$(26) \quad T_x^u = \left\{ y \in \mathbf{R}^m : y = \tilde{u}(x) + \left\langle \frac{Du}{|Du|}(x), z \right\rangle \text{ for some } z \in \mathbf{R}^n \right\}$$

39

40 ¹A multiset allows multiplicity of elements. Hence, $\{0, 01, 01\}$ is prefix free as a set, but
41 not as a multiset.

42

$\frac{1}{2}$ for every $x \in \Omega \setminus S_u$. Let $f: \mathbf{R}^m \rightarrow \mathbf{R}^k$ be a Lipschitz continuous function such
 $\frac{2}{2}$ that $f(0) = 0$, and let $v = f(u): \Omega \rightarrow \mathbf{R}^k$. Then $v \in BV(\Omega; \mathbf{R}^k)$ and

$$\frac{3}{4} \quad (27) \quad Jv = (f(u^+) - f(u^-)) \otimes \nu_u \cdot \mathcal{H}_{n-1}|_{S_u}.$$

$\frac{5}{6}$ In addition, for $|\widetilde{D}u|$ -almost every $x \in \Omega$ the restriction of the function f to
 $\frac{6}{7}$ T_x^u is differentiable at $\tilde{u}(x)$ and

$$\frac{8}{9} \quad (28) \quad \widetilde{D}v = \nabla(f|_{T_x^u})(\tilde{u}) \frac{\widetilde{D}u}{|\widetilde{D}u|} \cdot |\widetilde{D}u|.$$

$\frac{11}{12}$ Before proving the theorem, we state without proof three elementary remarks which will be useful in the sequel.

$\frac{13}{14}$ *Remark 7.1.* Let $\omega:]0, +\infty[\rightarrow]0, +\infty[$ be a continuous function such
 $\frac{14}{15}$ that $\omega(t) \rightarrow 0$ as $t \rightarrow 0$. Then

$$\frac{16}{17} \quad \lim_{h \rightarrow 0^+} g(\omega(h)) = L \Leftrightarrow \lim_{h \rightarrow 0^+} g(h) = L$$

$\frac{18}{18}$ for any function $g:]0, +\infty[\rightarrow \mathbf{R}$.

$\frac{19}{20}$ *Remark 7.2.* Let $g: \mathbf{R}^n \rightarrow \mathbf{R}$ be a Lipschitz continuous function and assume that

$$\frac{21}{22} \quad L(z) = \lim_{h \rightarrow 0^+} \frac{g(hz) - g(0)}{h}$$

$\frac{23}{24}$ exists for every $z \in \mathbf{Q}^n$ and that L is a linear function of z . Then g is differentiable at 0.

$\frac{26}{27}$ *Remark 7.3.* Let $A: \mathbf{R}^n \rightarrow \mathbf{R}^m$ be a linear function, and let $f: \mathbf{R}^m \rightarrow \mathbf{R}$
 $\frac{27}{28}$ be a function. Then the restriction of f to the range of A is differentiable at 0
 $\frac{28}{28}$ if and only if $f(A): \mathbf{R}^n \rightarrow \mathbf{R}$ is differentiable at 0 and

$$\frac{29}{30} \quad \nabla(f|_{\text{Im}(A)})(0)A = \nabla(f(A))(0).$$

$\frac{31}{32}$ *Proof.* We begin by showing that $v \in BV(\Omega; \mathbf{R}^k)$ and

$$\frac{33}{33} \quad (29) \quad |Dv|(B) \leq K |Du|(B) \quad \forall B \in \mathbf{B}(\Omega),$$

$\frac{34}{35}$ where $K > 0$ is the Lipschitz constant of f . By (13) and by the approximation result quoted in §3, it is possible to find a sequence $(u_h) \subset C^1(\Omega; \mathbf{R}^m)$
 $\frac{36}{37}$ converging to u in $L^1(\Omega; \mathbf{R}^m)$ and such that

$$\frac{38}{39} \quad \lim_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h| \, dx = |Du|(\Omega).$$

$\frac{40}{41}$ The functions $v_h = f(u_h)$ are locally Lipschitz continuous in Ω , and the definition of differential implies that $|\nabla v_h| \leq K |\nabla u_h|$ almost everywhere in Ω . The
 $\frac{41}{42}$

1 lower semicontinuity of the total variation and (13) yield

$$\begin{aligned}
\frac{2}{3} \quad & |Dv|(\Omega) \leq \liminf_{h \rightarrow +\infty} |Dv_h|(\Omega) = \liminf_{h \rightarrow +\infty} \int_{\Omega} |\nabla v_h| \, dx \\
\frac{4}{5} \quad (30) \quad & \leq K \liminf_{h \rightarrow +\infty} \int_{\Omega} |\nabla u_h| \, dx = K |Du|(\Omega).
\end{aligned}$$

6 Since $f(0) = 0$, we have also

$$\frac{8}{9} \quad \int_{\Omega} |v| \, dx \leq K \int_{\Omega} |u| \, dx;$$

11 therefore $u \in BV(\Omega; \mathbf{R}^k)$. Repeating the same argument for every open set
12 $A \subset \Omega$, we get (29) for every $B \in \mathbf{B}(\Omega)$, because $|Dv|$, $|Du|$ are Radon mea-
13 sures. To prove Lemma 6.1, first we observe that

$$\frac{14}{15} \quad (31) \quad S_v \subset S_u, \quad \tilde{v}(x) = f(\tilde{u}(x)) \quad \forall x \in \Omega \setminus S_u.$$

16 In fact, for every $\varepsilon > 0$ we have

$$\frac{17}{18} \quad \{y \in B_{\rho}(x) : |v(y) - f(\tilde{u}(x))| > \varepsilon\} \subset \{y \in B_{\rho}(x) : |u(y) - \tilde{u}(x)| > \varepsilon/K\},$$

19 hence

$$\frac{20}{21} \quad \lim_{\rho \rightarrow 0^+} \frac{|\{y \in B_{\rho}(x) : |v(y) - f(\tilde{u}(x))| > \varepsilon\}|}{\rho^n} = 0$$

22 whenever $x \in \Omega \setminus S_u$. By a similar argument, if $x \in S_u$ is a point such that
23 there exists a triplet (u^+, u^-, ν_u) satisfying (14), (15), then

$$\frac{25}{26} \quad (v^+(x) - v^-(x)) \otimes \nu_v = (f(u^+(x)) - f(u^-(x))) \otimes \nu_u \quad \text{if } x \in S_v$$

27 and $f(u^-(x)) = f(u^+(x))$ if $x \in S_u \setminus S_v$. Hence, by (1.8) we get

$$\begin{aligned}
\frac{28}{29} \quad Jv(B) &= \int_{B \cap S_v} (v^+ - v^-) \otimes \nu_v \, d\mathcal{H}_{n-1} = \int_{B \cap S_v} (f(u^+) - f(u^-)) \otimes \nu_u \, d\mathcal{H}_{n-1} \\
\frac{30}{31} \quad &= \int_{B \cap S_u} (f(u^+) - f(u^-)) \otimes \nu_u \, d\mathcal{H}_{n-1}
\end{aligned}$$

32 and Lemma 6.1 is proved. \square

34 To prove (31), it is not restrictive to assume that $k = 1$. Moreover, to
35 simplify our notation, from now on we shall assume that $\Omega = \mathbf{R}^n$. The proof
36 of (31) is divided into two steps. In the first step we prove the statement in
37 the one-dimensional case ($n = 1$), using Theorem 5.2. In the second step we
38 achieve the general result using Theorem 7.1.

39 *Step 1.* Assume that $n = 1$. Since S_u is at most countable, (7) yields
41 that $|\tilde{D}v|(S_u \setminus S_v) = 0$, so that (19) and (21) imply that $Dv = \tilde{D}v + Jv$ is the
42

1 Radon-Nikodým decomposition of Dv in absolutely continuous and singular
2 part with respect to $|\widetilde{Du}|$. By Theorem 5.2, we have

$$\begin{aligned} \frac{\widetilde{Dv}}{|\widetilde{Du}|}(t) &= \lim_{s \rightarrow t^+} \frac{Dv([t, s])}{|\widetilde{Du}|([t, s])}, & \frac{\widetilde{Du}}{|\widetilde{Du}|}(t) &= \lim_{s \rightarrow t^+} \frac{Du([t, s])}{|\widetilde{Du}|([t, s])} \end{aligned}$$

3
4
5
6 $|\widetilde{Du}|$ -almost everywhere in \mathbf{R} . It is well known (see, for instance, [20, 2.5.16])
7 that every one-dimensional function of bounded variation w has a unique left
8 continuous representative, i.e., a function \hat{w} such that $\hat{w} = w$ almost every-
9 where and $\lim_{s \rightarrow t^-} \hat{w}(s) = \hat{w}(t)$ for every $t \in \mathbf{R}$. These conditions imply

$$\text{(32)} \quad \hat{v}(t) = Du([-\infty, t]), \quad \hat{v}(t) = Dv([-\infty, t]) \quad \forall t \in \mathbf{R}$$

10
11 and

$$\text{(33)} \quad \hat{v}(t) = f(\hat{u}(t)) \quad \forall t \in \mathbf{R}.$$

12
13
14 Let $t \in \mathbf{R}$ be such that $|\widetilde{Du}|([t, s]) > 0$ for every $s > t$ and assume that the
15 limits in (22) exist. By (23) and (24) we get

$$\begin{aligned} \frac{\hat{v}(s) - \hat{v}(t)}{|\widetilde{Du}|([t, s])} &= \frac{f(\hat{u}(s)) - f(\hat{u}(t))}{|\widetilde{Du}|([t, s])} \\ &= \frac{f(\hat{u}(s)) - f(\hat{u}(t) + \frac{\widetilde{Du}}{|\widetilde{Du}|}(t) |\widetilde{Du}|([t, s]))}{|\widetilde{Du}|([t, s])} \\ &\quad + \frac{f(\hat{u}(t) + \frac{\widetilde{Du}}{|\widetilde{Du}|}(t) |\widetilde{Du}|([t, s])) - f(\hat{u}(t))}{|\widetilde{Du}|([t, s])} \end{aligned}$$

16
17 for every $s > t$. Using the Lipschitz condition on f we find

$$\begin{aligned} &\left| \frac{\hat{v}(s) - \hat{v}(t)}{|\widetilde{Du}|([t, s])} - \frac{f(\hat{u}(t) + \frac{\widetilde{Du}}{|\widetilde{Du}|}(t) |\widetilde{Du}|([t, s])) - f(\hat{u}(t))}{|\widetilde{Du}|([t, s])} \right| \\ &\leq K \left| \frac{\hat{u}(s) - \hat{u}(t)}{|\widetilde{Du}|([t, s])} - \frac{\widetilde{Du}}{|\widetilde{Du}|}(t) \right|. \end{aligned}$$

$\frac{1}{2}$ By (29), the function $s \rightarrow \left| \widetilde{D}u \right| ([t, s])$ is continuous and converges to 0 as $s \downarrow t$.
 $\frac{2}{2}$ Therefore Remark 7.1 and the previous inequality imply

$$\frac{3}{4} \quad \frac{\widetilde{D}v}{\left| \widetilde{D}u \right|}(t) = \lim_{h \rightarrow 0^+} \frac{f(\hat{u}(t) + h \frac{\widetilde{D}u}{\left| \widetilde{D}u \right|}(t)) - f(\hat{u}(t))}{h} \quad \left| \widetilde{D}u \right| \text{-a.e. in } \mathbf{R}.$$

$\frac{8}{9}$ By (22), $\hat{u}(x) = \tilde{u}(x)$ for every $x \in \mathbf{R} \setminus S_u$; moreover, applying the same argument to the functions $u'(t) = u(-t)$, $v'(t) = f(u'(t)) = v(-t)$, we get

$$\frac{11}{14} \quad \frac{\widetilde{D}v}{\left| \widetilde{D}u \right|}(t) = \lim_{h \rightarrow 0} \frac{f(\tilde{u}(t) + h \frac{\widetilde{D}u}{\left| \widetilde{D}u \right|}(t)) - f(\tilde{u}(t))}{h} \quad \left| \widetilde{D}u \right| \text{-a.e. in } \mathbf{R}$$

$\frac{15}{15}$ and our statement is proved.

$\frac{16}{17}$ *Step 2.* Let us consider now the general case $n > 1$. Let $\nu \in \mathbf{R}^n$ be such
 $\frac{17}{18}$ that $|\nu| = 1$, and let $\pi_\nu = \{y \in \mathbf{R}^n : \langle y, \nu \rangle = 0\}$. In the following, we shall
 $\frac{18}{19}$ identify \mathbf{R}^n with $\pi_\nu \times \mathbf{R}$, and we shall denote by y the variable ranging in π_ν
 $\frac{19}{20}$ and by t the variable ranging in \mathbf{R} . By the just proven one-dimensional result,
 $\frac{20}{21}$ and by Theorem 3.3, we get

$$\frac{22}{25} \quad \lim_{h \rightarrow 0} \frac{f(\tilde{u}(y + t\nu) + h \frac{\widetilde{D}u_y}{\left| \widetilde{D}u_y \right|}(t)) - f(\tilde{u}(y + t\nu))}{h} = \frac{\widetilde{D}v_y}{\left| \widetilde{D}u_y \right|}(t) \quad \left| \widetilde{D}u_y \right| \text{-a.e. in } \mathbf{R}$$

$\frac{26}{27}$ for \mathcal{H}_{n-1} -almost every $y \in \pi_\nu$. We claim that

$$\frac{28}{29} \quad (34) \quad \frac{\langle \widetilde{D}u, \nu \rangle}{\left| \langle \widetilde{D}u, \nu \rangle \right|}(y + t\nu) = \frac{\widetilde{D}u_y}{\left| \widetilde{D}u_y \right|}(t) \quad \left| \widetilde{D}u_y \right| \text{-a.e. in } \mathbf{R}$$

$\frac{31}{31}$ for \mathcal{H}_{n-1} -almost every $y \in \pi_\nu$. In fact, by (16) and (18) we get

$$\begin{aligned} \frac{32}{33} \quad & \int_{\pi_\nu} \frac{\widetilde{D}u_y}{\left| \widetilde{D}u_y \right|} \cdot \left| \widetilde{D}u_y \right| d\mathcal{H}_{n-1}(y) = \int_{\pi_\nu} \widetilde{D}u_y d\mathcal{H}_{n-1}(y) \\ \frac{34}{35} \quad & = \langle \widetilde{D}u, \nu \rangle = \frac{\langle \widetilde{D}u, \nu \rangle}{\left| \langle \widetilde{D}u, \nu \rangle \right|} \cdot \left| \langle \widetilde{D}u, \nu \rangle \right| = \int_{\pi_\nu} \frac{\langle \widetilde{D}u, \nu \rangle}{\left| \langle \widetilde{D}u, \nu \rangle \right|}(y + \cdot\nu) \cdot \left| \widetilde{D}u_y \right| d\mathcal{H}_{n-1}(y) \end{aligned}$$

$\frac{38}{39}$ and (24) follows from (13). By the same argument it is possible to prove that

$$\frac{40}{42} \quad (35) \quad \frac{\langle \widetilde{D}v, \nu \rangle}{\left| \langle \widetilde{D}u, \nu \rangle \right|}(y + t\nu) = \frac{\widetilde{D}v_y}{\left| \widetilde{D}u_y \right|}(t) \quad \left| \widetilde{D}u_y \right| \text{-a.e. in } \mathbf{R}$$

1 for \mathcal{H}_{n-1} -almost every $y \in \pi_\nu$. By (24) and (25) we get

$$\begin{aligned} & \frac{f(\tilde{u}(y + t\nu) + h \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(y + t\nu)) - f(\tilde{u}(y + t\nu))}{h} \\ & \lim_{h \rightarrow 0} \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(y + t\nu) \end{aligned}$$

7 for \mathcal{H}_{n-1} -almost every $y \in \pi_\nu$, and using again (14), (15) we get

$$\begin{aligned} & \frac{f(\tilde{u}(x) + h \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)) - f(\tilde{u}(x))}{h} \\ & \lim_{h \rightarrow 0} \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x) \end{aligned}$$

13 $|\langle \tilde{D}u, \nu \rangle|$ -a.e. in \mathbf{R}^n .

15 Since the function $|\langle \tilde{D}u, \nu \rangle| / |\tilde{D}u|$ is strictly positive $|\langle \tilde{D}u, \nu \rangle|$ -almost ev-
16 erywhere, we obtain also

$$\begin{aligned} & \frac{f(\tilde{u}(x) + h \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|}(x) \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x)) - f(\tilde{u}(x))}{h} \\ & \lim_{h \rightarrow 0} \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|}(x) \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|}(x) \end{aligned}$$

25 $|\langle \tilde{D}u, \nu \rangle|$ -almost everywhere in \mathbf{R}^n .

27 Finally, since

$$\begin{aligned} & \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|} \frac{\langle \tilde{D}u, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|} = \frac{\langle \tilde{D}u, \nu \rangle}{|\tilde{D}u|} = \left\langle \frac{\tilde{D}u}{|\tilde{D}u|}, \nu \right\rangle \quad |\tilde{D}u| \text{-a.e. in } \mathbf{R}^n \\ & \frac{|\langle \tilde{D}u, \nu \rangle|}{|\tilde{D}u|} \frac{\langle \tilde{D}v, \nu \rangle}{|\langle \tilde{D}u, \nu \rangle|} = \frac{\langle \tilde{D}v, \nu \rangle}{|\tilde{D}u|} = \left\langle \frac{\tilde{D}v}{|\tilde{D}u|}, \nu \right\rangle \quad |\tilde{D}u| \text{-a.e. in } \mathbf{R}^n \end{aligned}$$

35 and since both sides of (33) are zero $|\tilde{D}u|$ -almost everywhere on $|\langle \tilde{D}u, \nu \rangle|$ -
36 negligible sets, we conclude that

$$\begin{aligned} & \frac{f\left(\tilde{u}(x) + h \left\langle \frac{\tilde{D}u}{|\tilde{D}u|}(x), \nu \right\rangle\right) - f(\tilde{u}(x))}{h} \\ & \lim_{h \rightarrow 0} \left\langle \frac{\tilde{D}v}{|\tilde{D}u|}(x), \nu \right\rangle, \end{aligned}$$

$\left|\widetilde{D}u\right|$ -a.e. in \mathbf{R}^n . Since ν is arbitrary, by Remarks 7.2 and 7.3 the restriction of f to the affine space T_x^u is differentiable at $\tilde{u}(x)$ for $\left|\widetilde{D}u\right|$ -almost every $x \in \mathbf{R}^n$ and (26) holds. \square

It follows from (13), (14), and (15) that

$$(36) \quad D(t_1, \dots, t_n) = \sum_{I \in \mathbf{n}} (-1)^{|I|-1} |I| \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}).$$

Let $t_i = \hat{x}_i$, $i = 1, \dots, n$. Lemma 1 leads to

$$(37) \quad D(\hat{x}_1, \dots, \hat{x}_n) = \prod_{i \in \mathbf{n}} \hat{x}_i \sum_{I \in \mathbf{n}} (-1)^{|I|-1} |I| \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I}|\bar{I}).$$

By (3), (13), and (37), we have the following result:

THEOREM 7.2.

$$(38) \quad H_c = \frac{1}{2n} \sum_{l=1}^n l (-1)^{l-1} A_l^{(\lambda)},$$

where

$$(39) \quad A_l^{(\lambda)} = \sum_{I_l \subseteq \mathbf{n}} \text{per } \mathbf{A}^{(\lambda)}(I_l|I_l) \det \mathbf{A}^{(\lambda)}(\bar{I}_l|\bar{I}_l), |I_l| = l.$$

It is worth noting that $A_l^{(\lambda)}$ of (39) is similar to the coefficients b_l of the characteristic polynomial of (10). It is well known in graph theory that the coefficients b_l can be expressed as a sum over certain subgraphs. It is interesting to see whether A_l , $\lambda = 0$, structural properties of a graph.

We may call (38) a parametric representation of H_c . In computation, the parameter λ_i plays very important roles. The choice of the parameter usually depends on the properties of the given graph. For a complete graph K_n , let $\lambda_i = 1$, $i = 1, \dots, n$. It follows from (39) that

$$(40) \quad A_l^{(1)} = \begin{cases} n!, & \text{if } l = 1 \\ 0, & \text{otherwise.} \end{cases}$$

By (38)

$$(41) \quad H_c = \frac{1}{2}(n-1)!.$$

For a complete bipartite graph $K_{n_1 n_2}$, let $\lambda_i = 0$, $i = 1, \dots, n$. By (39),

$$(42) \quad A_l = \begin{cases} -n_1! n_2! \delta_{n_1 n_2}, & \text{if } l = 2 \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 7.2 leads to

$$(43) \quad H_c = \frac{1}{n_1 + n_2} n_1! n_2! \delta_{n_1 n_2}.$$

Now, we consider an asymmetrical approach. Theorem 3.3 leads to

$$(44) \quad \det \mathbf{K}(t = 1, t_1, \dots, t_n; l|l) \\ = \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \prod_{i \in I} t_i \prod_{j \in I} (D_j + \lambda_j t_j) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\}).$$

By (3) and (16) we have the following asymmetrical result:

THEOREM 7.3.

$$(45) \quad H_c = \frac{1}{2} \sum_{I \subseteq \mathbf{n} - \{l\}} (-1)^{|I|} \text{per } \mathbf{A}^{(\lambda)}(I|I) \det \mathbf{A}^{(\lambda)}(\bar{I} \cup \{l\} | \bar{I} \cup \{l\})$$

which reduces to Goulden–Jackson’s formula when $\lambda_i = 0, i = 1, \dots, n$ [14].

8. Various font features of the amsmath package

8.1. *Bold versions of special symbols.* In the amsmath package `\boldsymbol` is used for getting individual bold math symbols and bold Greek letters—everything in math except for letters of the Latin alphabet, where you’d use `\mathbf`. For example,

```
A_\infty + \pi A_0 \sim
\mathbf{A}_{\boldsymbol{\infty}} \boldsymbol{+}
\boldsymbol{\pi} \mathbf{A}_{\boldsymbol{0}}
```

looks like this:

$$A_\infty + \pi A_0 \sim \mathbf{A}_\infty + \pi \mathbf{A}_0$$

8.2. *“Poor man’s bold”.* If a bold version of a particular symbol doesn’t exist in the available fonts, then `\boldsymbol` can’t be used to make that symbol bold. At the present time, this means that `\boldsymbol` can’t be used with symbols from the `msam` and `msbm` fonts, among others. In some cases, poor man’s bold (`\pmb`) can be used instead of `\boldsymbol`:

```
\frac{\partial x}{\partial y} \bigg|_{\partial z}
\pmb{\bigg|_{\partial z}}
\frac{\partial y}{\partial z}
```

So-called “large operator” symbols such as \sum and \prod require an additional command, `\mathop`, to produce proper spacing and limits when `\pmb` is used. For further details see *The T_EXbook*.

$$\sum_{\substack{i < B \\ i \text{ odd}}} \prod_{\kappa} \kappa F(r_i) \quad \sum_{\substack{i < B \\ i \text{ odd}}} \prod_{\kappa} \kappa(r_i)$$

```

1  \[\sum_{\substack{i<B\\\text{\$i\$ odd}}}]
2  \prod_{\kappa} \kappa F(r_i)\quad
3  \mathop{\pmb{\sum}}_{\substack{i<B\\\text{\$i\$ odd}}}
4  \mathop{\pmb{\prod}}_{\kappa} \kappa(r_i)
5  \]

```

6

7

9. Compound symbols and other features

8

9

10

11

9.1. *Multiple integral signs.* `\iint`, `\iiint`, and `\iiiiint` give multiple integral signs with the spacing between them nicely adjusted, in both text and display style. `\idotsint` gives two integral signs with dots between them.

12

13

14

15

16

$$(46) \quad \iint_A f(x, y) \, dx \, dy \quad \iiint_A f(x, y, z) \, dx \, dy \, dz$$

$$(47) \quad \iiidotsint_A f(w, x, y, z) \, dw \, dx \, dy \, dz \quad \int \cdots \int_A f(x_1, \dots, x_k)$$

17

18

19

20

21

22

23

24

25

9.2. *Over and under arrows.* Some extra over and under arrow operations are provided in the `amsmath` package. (Basic \LaTeX provides `\overrightarrow` and `\overleftarrow`).

$$\begin{aligned} \overrightarrow{\psi_\delta(t) E_t h} &= \psi_\delta(t) E_t h \\ \overleftarrow{\psi_\delta(t) E_t h} &= \psi_\delta(t) E_t h \\ \overleftrightarrow{\psi_\delta(t) E_t h} &= \psi_\delta(t) E_t h \end{aligned}$$

26 `\begin{align*}`27 `\overrightarrow{\psi_\delta(t) E_t h}&`28 `=\underrightarrow{\psi_\delta(t) E_t h}\`29 `\overleftarrow{\psi_\delta(t) E_t h}&`30 `=\underleftarrow{\psi_\delta(t) E_t h}\`31 `\overleftrightharrow{\psi_\delta(t) E_t h}&`32 `=\underleftrightharrow{\psi_\delta(t) E_t h}`33 `\end{align*}`

34 These all scale properly in subscript sizes:

35

36

37

38

$$\int_{\overrightarrow{AB}} ax \, dx$$

```

\[\int_{\overrightarrow{AB}} ax \, dx\]

```

39

40

41

42

9.3. *Dots.* Normally you need only type `\dots` for ellipsis dots in a math formula. The main exception is when the dots fall at the end of the formula; then you need to specify one of `\dotsc` (series dots, after a comma), `\dotscb`

1 (binary dots, for binary relations or operators), `\dotsm` (multiplication dots),
2 or `\dotsi` (dots after an integral). For example, the input

3 Then we have the series `\$A_1,A_2,\dotscsc\$,`
4 the regional sum `\$A_1+A_2+\dotsb\$,`
5 the orthogonal product `\$A_1A_2\dotscsm\$,`
6 and the infinite integral
7 `\[\int_{A_1}\int_{A_2}\dotsi\]`.

8 produces

9 Then we have the series A_1, A_2, \dots , the regional sum $A_1 + A_2 +$
10 \dots , the orthogonal product $A_1 A_2 \dots$, and the infinite integral

11
12
$$\int_{A_1} \int_{A_2} \dots$$

13
14 9.4. *Accents in math.* Double accents:

15 $\hat{H} \ \check{C} \ \tilde{T} \ \acute{A} \ \grave{G} \ \dot{D} \ \ddot{D} \ \breve{B} \ \bar{B} \ \vec{V}$

16
17 `\[\Hat{\Hat{H}}\quad\Check{\Check{C}}\quad`
18 `\Tilde{\Tilde{T}}\quad\Acute{\Acute{A}}\quad`
19 `\Grave{\Grave{G}}\quad\Dot{\Dot{D}}\quad`
20 `\Ddot{\Ddot{D}}\quad\Breve{\Breve{B}}\quad`
21 `\Bar{\Bar{B}}\quad\Vec{\Vec{V}}\]`

22 This double accent operation is complicated and tends to slow down the pro-
23 cessing of a \LaTeX file.

24
25 9.5. *Dot accents.* `\dddots` and `\ddddots` are available to produce triple and
26 quadruple dot accents in addition to the `\dot` and `\ddot` accents already avail-
27 able in \LaTeX :

28 $\ddot{Q} \quad \dddot{R}$

29 `\[\dddots{Q}\quad\ddddots{R}\]`

30
31 9.6. *Roots.* In the `amsmath` package `\leftroot` and `\uproot` allow you to
32 adjust the position of the root index of a radical:

33 `\sqrt[\leftroot{-2}\uproot{2}\beta]{k}`

34 gives good positioning of the β :

35
$$\sqrt[\beta]{k}$$

36
37 9.7. *Boxed formulas.* The command `\boxed` puts a box around its argu-
38 ment, like `\fbox` except that the contents are in math mode:

39 `\boxed{W_t - F \subseteq V(P_i) \subseteq W_t}`

40
$$\boxed{W_t - F \subseteq V(P_i) \subseteq W_t}.$$

41
42

9.8. *Extensible arrows.* `\xleftarrow` and `\xrightarrow` produce arrows that extend automatically to accommodate unusually wide subscripts or superscripts. The text of the subscript or superscript are given as an optional resp. mandatory argument: Example:

$$0 \underset{\zeta}{\xleftarrow{\alpha}} F \times \Delta[n-1] \xrightarrow{\partial_0 \alpha(b)} E^{\partial_0 b}$$

```

\left[0 \xleftarrow[\zeta]{\alpha} F \times \triangle[n-1]
\xrightarrow{\partial_0 \alpha(b)} E^{\partial_0 b}\right]

```

9.9. `\overset`, `\underset`, and `\sideset`. Examples:

$$\overset{*}{X} \quad X_* \quad \overset{a}{\underset{b}{X}}$$

```

\left[\overset{*}{X} \quad X_* \quad \overset{a}{\underset{b}{X}}\right]

```

The command `\sideset` is for a rather special purpose: putting symbols at the subscript and superscript corners of a large operator symbol such as \sum or \prod , without affecting the placement of limits. Examples:

$$\overset{*}{\underset{*}{\prod}}_k \quad \sum'_{0 \leq i \leq m} E_i \beta x$$

```

\left[\overset{*}{\underset{*}{\prod}}_k \quad \sum'_{0 \leq i \leq m} E_i \beta x\right]

```

9.10. *The `\text` command.* The main use of the command `\text` is for words or phrases in a display:

$$\mathbf{y} = \mathbf{y}' \quad \text{if and only if} \quad y'_k = \delta_k y_{\tau(k)}$$

```

\left[\mathbf{y} = \mathbf{y}' \quad \text{if and only if} \quad y'_k = \delta_k y_{\tau(k)}\right]

```

9.11. *Operator names.* The more common math functions such as `\log`, `\sin`, `\lim` and `\lim` have predefined control sequences: `\log`, `\sin`, `\lim`. The `amsmath` package provides `\DeclareMathOperator` and `\DeclareMathOperator*` for producing new function names that will have the same typographical treatment. Examples:

$$\|f\|_{\infty} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|$$

```

\left[\|f\|_{\infty} = \text{ess sup}_{x \in \mathbb{R}^n} |f(x)|\right]

```

$$\text{meas}_1 \{u \in \mathbb{R}_+^1 : f^*(u) > \alpha\} = \text{meas}_n \{x \in \mathbb{R}^n : |f(x)| \geq \alpha\} \quad \forall \alpha > 0.$$

1 $\backslash[\backslash\text{meas}_1\{u\in R_{+}^1\colon f^*(u)>\alpha\}$
 2 $=\backslash\text{meas}_n\{x\in R^n\colon \text{abs}\{f(x)\}\geq\alpha\}$
 3 $\backslash\text{quad } \backslash\text{forall}\alpha>0.\backslash]$

4 $\backslash\text{esssup}$ and $\backslash\text{meas}$ would be defined in the document preamble as

5 $\backslash\text{DeclareMathOperator}\{\backslash\text{esssup}\}\{\text{ess}\backslash,\text{sup}\}$

6 $\backslash\text{DeclareMathOperator}\{\backslash\text{meas}\}\{\text{meas}\}$

7
 8 The following special operator names are predefined in the `amsmath` pack-
 9 age: $\backslash\text{varlimsup}$, $\backslash\text{varliminf}$, $\backslash\text{varinjlim}$, and $\backslash\text{varprojlim}$. Here's what
 10 they look like in use:

11 (48)
$$\overline{\lim}_{n\rightarrow\infty} Q(u_n, u_n - u^\#) \leq 0$$

12 (49)
$$\underline{\lim}_{n\rightarrow\infty} |a_{n+1}| / |a_n| = 0$$

13 (50)
$$\varinjlim (m_i^\lambda \cdot)^* \leq 0$$

14 (51)
$$\varprojlim_{p\in S(A)} A_p \leq 0$$

15 $\backslash\text{begin}\{\text{align}\}$

16 $\&\backslash\text{varlimsup}_{n\rightarrow\infty}$

17 $\backslash\text{mathcal}\{Q\}(u_n, u_n - u^\#)\leq 0\backslash\backslash$

18 $\&\backslash\text{varliminf}_{n\rightarrow\infty}$

19 $\backslash\left\lvert a_{n+1}\right\rvert/\backslash\right\rvert a_n\backslash\right\rvert=0\backslash\backslash$

20 $\&\backslash\text{varinjlim } (m_i^\lambda \cdot)^*\leq 0\backslash\backslash$

21 $\&\backslash\text{varprojlim}_{p\in S(A)} A_p\leq 0$

22 $\backslash\text{end}\{\text{align}\}$

23
 24 9.12. *$\backslash\text{mod}$ and its relatives.* The commands $\backslash\text{mod}$ and $\backslash\text{pod}$ are variants
 25 of $\backslash\text{pmod}$ preferred by some authors; $\backslash\text{mod}$ omits the parentheses, whereas $\backslash\text{pod}$
 26 omits the ‘mod’ and retains the parentheses. Examples:

27 (52)
$$x \equiv y + 1 \pmod{m^2}$$

28 (53)
$$x \equiv y + 1 \text{ mod } m^2$$

29 (54)
$$x \equiv y + 1 \pmod{m^2}$$

30 $\backslash\text{begin}\{\text{align}\}$

31 $x\equiv y+1\pmod{m^2}\backslash\backslash$

32 $x\equiv y+1\text{mod}\{m^2}\backslash\backslash$

33 $x\equiv y+1\pmod{m^2}$

34 $\backslash\text{end}\{\text{align}\}$

35
 36 9.13. *Fractions and related constructions.* The usual notation for binomi-
 37 als is similar to the fraction concept, so it has a similar command $\backslash\text{binom}$ with
 38

39

1 two arguments. Example:

$$\begin{aligned}
 & \sum_{\gamma \in \Gamma_C} I_\gamma = 2^k - \binom{k}{1} 2^{k-1} + \binom{k}{2} 2^{k-2} \\
 & + \dots + (-1)^l \binom{k}{l} 2^{k-l} + \dots + (-1)^k \\
 & = (2-1)^k = 1
 \end{aligned}$$

```

2
3
4
5 (55)
6
7
8
9 \begin{equation}
10 \begin{split}
11 [\sum_{\gamma \in \Gamma_C} I_\gamma &
12 = 2^k - \binom{k}{1} 2^{k-1} + \binom{k}{2} 2^{k-2} \\
13 & \quad + \dots + (-1)^1 \binom{k}{1} 2^{k-1} \\
14 & \quad + \dots + (-1)^k \\
15 & \quad = (2-1)^k = 1 \\
16 \end{split}
17 \end{equation}

```

18 There are also abbreviations

```

19 \dfrac      \dbinom
20 \tfrac      \tbinom

```

21 for the commonly needed constructions

```

22 {\displaystyle\frac ... }  {\displaystyle\binom ... }
23 {\textstyle\frac ... }    {\textstyle\binom ... }

```

24 The generalized fraction command `\genfrac` provides full access to the
25 six T_EX fraction primitives:

$$\text{(56)} \quad \overline{\frac{n+1}{2}} \quad \overwithdelims: \left\langle \frac{n+1}{2} \right\rangle$$

$$\text{(57)} \quad \atop \frac{n+1}{2} \quad \atopwithdelims: \left(\frac{n+1}{2} \right)$$

$$\text{(58)} \quad \above \frac{n+1}{2} \quad \abovewithdelims: \left[\frac{n+1}{2} \right]$$

```

34 \text{\cn{over}: } & \genfrac{}{}{}{}{n+1}{2} &
35 \text{\cn{overwithdelims}: } &
36 \genfrac{\langle \rangle}{\langle \rangle}{}{}{n+1}{2} \\
37 \text{\cn{atop}: } & \genfrac{}{}{0pt}{}{n+1}{2} &
38 \text{\cn{atopwithdelims}: } &
39 \genfrac{()}{()}{0pt}{}{n+1}{2} \\
40 \text{\cn{above}: } & \genfrac{}{}{1pt}{}{n+1}{2} &
41 \text{\cn{abovewithdelims}: } &

```

42

`\genfrac{[]{}{1pt}{}{n+1}{2}`

9.14. *Continued fractions.* The continued fraction

$$(59) \quad \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \frac{1}{\sqrt{2} + \dots}}}}}$$

can be obtained by typing

```
\frac{1}{\sqrt{2}+
\frac{1}{\sqrt{2}+
\frac{1}{\sqrt{2}+
\frac{1}{\sqrt{2}+
\frac{1}{\sqrt{2}+\dotsb
}}}}}
```

Left or right placement of any of the numerators is accomplished by using `\frac[l]` or `\frac[r]` instead of `\frac`.

9.15. *Smash.* In `amsmath` there are optional arguments `t` and `b` for the plain `TeX` command `\smash`, because sometimes it is advantageous to be able to ‘smash’ only the top or only the bottom of something while retaining the natural depth or height. In the formula $X_j = (1/\sqrt{\lambda_j})X'_j$ `\smash[b]` has been used to limit the size of the radical symbol.

```
 $X_j=(1/\sqrt{\smash[b]{\lambda_j}})X'_j$ 
```

Without the use of `\smash[b]` the formula would have appeared thus: $X_j = (1/\sqrt{\lambda_j})X'_j$, with the radical extending to encompass the depth of the subscript j .

9.16. *The ‘cases’ environment.* ‘Cases’ constructions like the following can be produced using the `cases` environment.

$$(60) \quad P_{r-j} = \begin{cases} 0 & \text{if } r-j \text{ is odd,} \\ r!(-1)^{(r-j)/2} & \text{if } r-j \text{ is even.} \end{cases}$$

```
\begin{equation} P_{r-j}=
\begin{cases}
0& \text{\text{if } $r-j$ is odd},\ \
r!\,(-1)^{(r-j)/2}& \text{\text{if } $r-j$ is even}.}
\end{cases}
\end{equation}
```

1 Notice the use of `\text` and the embedded math.

2 9.17. *Matrix*. Here are samples of the matrix environments, `\matrix`,
3 `\pmatrix`, `\bmatrix`, `\Bmatrix`, `\vmatrix` and `\Vmatrix`:
4

5

6

7

8

9

10

```

11 \begin{matrix}
12 \vartheta& \varrho\\ \varphi& \varpi\end{matrix}
13 \end{matrix}\quad
14 \begin{pmatrix}
15 \vartheta& \varrho\\ \varphi& \varpi\end{pmatrix}
16 \end{pmatrix}\quad
17 \begin{bmatrix}
18 \vartheta& \varrho\\ \varphi& \varpi\end{bmatrix}
19 \end{bmatrix}\quad
20 \begin{Bmatrix}
21 \vartheta& \varrho\\ \varphi& \varpi\end{Bmatrix}
22 \end{Bmatrix}\quad
23 \begin{vmatrix}
24 \vartheta& \varrho\\ \varphi& \varpi\end{vmatrix}
25 \end{vmatrix}\quad
26 \begin{Vmatrix}
27 \vartheta& \varrho\\ \varphi& \varpi\end{Vmatrix}
28 \end{Vmatrix}

```

29

30 To produce a small matrix suitable for use in text, use the `smallmatrix`
31 environment.

32

```

33 \begin{math}
34 \bigl( \begin{smallmatrix}
35 a&b\\ c&d\end{smallmatrix}
36 \end{smallmatrix} \biggr)
37 \end{math}

```

38

39 To show the effect of the matrix on the surrounding lines of a paragraph, we
40 put it here: $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and follow it with enough text to ensure that there will be
41 at least one full line below the matrix.
42

`\hdotsfor{number}` produces a row of dots in a matrix spanning the given number of columns:

$$W(\Phi) = \begin{vmatrix} \frac{\varphi}{(\varphi_1, \varepsilon_1)} & 0 & \dots & 0 \\ \frac{\varphi^{k_{n2}}}{(\varphi_2, \varepsilon_1)} & \frac{\varphi}{(\varphi_2, \varepsilon_2)} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ \frac{\varphi^{k_{n1}}}{(\varphi_n, \varepsilon_1)} & \frac{\varphi^{k_{n2}}}{(\varphi_n, \varepsilon_2)} & \dots & \frac{\varphi^{k_{nn-1}}}{(\varphi_n, \varepsilon_{n-1})} & \frac{\varphi}{(\varphi_n, \varepsilon_n)} \end{vmatrix}$$

```

10 \[W(\Phi)= \begin{vmatrix}
11 \dfrac{\varphi}{(\varphi_1, \varepsilon_1)}&0&\dots&0\
12 \dfrac{\varphi^{k_{n2}}}{(\varphi_2, \varepsilon_1)}&{\dfrac{\varphi}{(\varphi_2, \varepsilon_2)}}&\dots&0\
13 \dfrac{\varphi}{(\varphi_2, \varepsilon_2)}&\dots&\dots&0\
14 \hdotsfor{5}\
15 \dfrac{\varphi^{k_{n1}}}{(\varphi_n, \varepsilon_1)}&{\dfrac{\varphi^{k_{n2}}}{(\varphi_n, \varepsilon_2)}}&\dots&{\dfrac{\varphi^{k_{nn-1}}}{(\varphi_n, \varepsilon_{n-1})}}&{\dfrac{\varphi}{(\varphi_n, \varepsilon_n)}}\
16 \dfrac{\varphi^{k_{n2}}}{(\varphi_n, \varepsilon_2)}&\dots&\dots&\dots&\dots\
17 \dfrac{\varphi^{k_{n,n-1}}}{(\varphi_n, \varepsilon_{n-1})}&\dots&\dots&\dots&\dots\
18 \dfrac{\varphi}{(\varphi_n, \varepsilon_n)}\
19 \end{vmatrix}\]

```

The spacing of the dots can be varied through use of a square-bracket option, for example, `\hdotsfor[1.5]{3}`. The number in square brackets will be used as a multiplier; the normal value is 1.

9.18. *The `\substack` command.* The `\substack` command can be used to produce a multiline subscript or superscript: for example

```

26 \sum_{\substack{0 \leq i \leq m \\ 0 < j < n}} P(i, j)

```

produces a two-line subscript underneath the sum:

$$(62) \quad \sum_{\substack{0 \leq i \leq m \\ 0 < j < n}} P(i, j)$$

A slightly more generalized form is the `subarray` environment which allows you to specify that each line should be left-aligned instead of centered, as here:

Maybe "... as below"?

$$(63) \quad \sum_{\subarray{1} 0 \leq i \leq m \\ 0 < j < n} P(i, j)$$

```

38 \sum_{\begin{subarray}{l}
39 0 \leq i \leq m \\
40 \end{subarray}}
41 P(i, j)
42

```

1 9.19. *Big-g-g delimiters.* Here are some big delimiters, first in `\normalsize`:

2
$$\left(\mathbf{E}_y \int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) ds \right)$$

3

4 `\[\biggl(\mathbf{E}_{-y}`

5 `\int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) \, ds`

6 `\biggr)`

7 `\]`

8

9 and now in `\Large` size:

10
$$\left(\mathbf{E}_y \int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) ds \right)$$

11

12

13 `\{\Large`

14 `\[\biggl(\mathbf{E}_{-y}`

15 `\int_0^{t_\varepsilon} L_{x,y^x(s)} \varphi(x) \, ds`

16 `\biggr)`

17 `\}\}`

18

19 9.20. *Acknowledgements.* The authors are grateful to NASA (grant 123456)

20 and NIH. They acknowledge the generous help of other agencies .

21

22

References

23

24

25

26

27

28

29

30

31

32

33

34

35

36

37

38

39

40

41

42

- [1] V. I. ARNOLD, *Mathematical Methods of Classical Mechanics*, second ed., *Graduate Texts in Mathematics* **60**, Springer, New York, 1989.
- [2] W. DIFFIE and E. HELLMAN, New directions in cryptography, *IEEE Transactions on Information Theory* **22** no. 5 (1976), 644–654.
- [3] D. H. FREMLIN, Cichon’s diagram, presented at the Séminaire Initiation à l’Analyse, G. Choquet, M. Rogalski, J. Saint Raymond, at the Université Pierre et Marie Curie, Paris, 23e année., 1983/194.
- [4] D. H. FREMLIN, *Topological Riesz Spaces and Measure Theory*, Cambridge University Press, 2008.
- [5] I. P. GOULDEN and D. M. JACKSON, The enumeration of directed closed Euler trails and directed Hamiltonian circuits by Langrangian methods, *European Journal of Combinatorics* **2** (1981), 131–212.
- [6] C. DE GROOT, D. WÜRTZ, M. HANF, R. PEIKERT, T. KOLLER, and K. H. HOFFMANN, Stochastic optimization—efficient algorithms to solve complex problems, in *System Modelling and Optimization, Proceedings of the Fifteenth IFIP Conference (Zürich)* (P. KALL, ed.), Springer-Verlag, 1992, pp. 546–555.
- [7] F. HARARY and E. M. PALMER, *Graphical Enumeration*, Academic Press, 1973.
- [8] R. IMPAGLIAZZO, L. LEVIN, and M. LUBY, Pseudo-random generation from one-way functions, in *Proc. 21st STOC (Seattle, WA, USA)*, ACM, New York, 1989, pp. 12–24.

- 1 [9] D. E. KNUTH, *The T_EXbook*, with illustrations by Duane Bibby, *Computers &*
2 *Typesetting A*, Addison-Wesley Publishing Company, Reading, MA, 1994.
- 3 [10] M. KOJIMA, S. MIZUNO, and A. YOSHISE, *A New Continuation Method for Com-*
4 *plementarity Problems With Uniform p-Functions*, Tech. Report B-194, Tokyo
Inst. of Technology, Dept. of Information Sciences, Tokyo, 1987.
- 5 [11] M. KOJIMA, S. MIZUNO, and A. YOSHISE, *A Polynomial-Time Algorithm For a*
6 *Class of Linear Complementarity Problems*, Tech. Report B-193, Tokyo Inst. of
Technology, Dept. of Information Sciences, Tokyo, 1987.
- 7 [12] H. W. LENSTRA, JR. and F. OORT, Simple abelian varieties having a prescribed
8 formal isogeny type., *J. Pure Appl. Algebra* **4** (1974), 47–53. [MR 0279.14009](#).
[Zbl 50:7163](#). [https://doi.org/10.1016/0022-4049\(74\)90029-2](https://doi.org/10.1016/0022-4049(74)90029-2).
- 9 [13] C. J. LIU and Y. CHOW, On operator and formal sum methods for graph enu-
10 meration problems, *SIAM Journal of Algorithms and Discrete Methods* **5** (1984),
11 384–438.
- 12 [14] M. MARCUS and H. MINC, A survey of matrix theory and matrix inequalities,
13 *Complementary Series in Mathematics* **14** (1964), 21–48.
- 14 [15] A. D. MICHAL, Differential calculus in linear topological spaces, *Proc. nat. Acad.*
15 *Sci. USA* **24** (1938), 340–342. [JFM 64.0366.02](#).
- 16 [16] A. D. MICHAL, *Matrix and Tensor Calculus*, *GALCIT Aeronautical Series*, John
17 Wiley & Sons, Inc.; Chapman & Hall, Ltd., New York; London, 1948.
- 18 [17] A. MINASYAN and D. OSIN, *Normal Automorphisms of Relatively Hyperbolic*
19 *Groups*, 2008. [arXiv 0809.2408](#).
- 20 [18] S. MIZUNO, A. YOSHISE, and T. KIKUCHI, *Practical Polynomial Time Algo-*
21 *rithms for Linear Complementarity Problems*, Tech. Report 13, Tokyo Inst. of
22 Technology, Dept. of Industrial Engineering and Management, Tokyo, April 1988.
- 23 [19] R. D. MONTEIRO and I. ADLER, *Interior Path Following Primal-Dual Algo-*
24 *rithms, Part II: Quadratic Programming*, Working paper, Dept. of Industrial En-
25 gineering and Operations Research, August 1987.
- 26 [20] E. M. STEIN, *Singular Integrals and Differentiability Properties of Functions*,
27 Princeton Univ. Press, Princeton, NJ, 1970.
- 28 [21] Y. YE, *Interior Algorithms for Linear, Quadratic and Linearly Constrained Con-*
29 *convex Programming*, Ph.D. thesis, Stanford Univ., Dept. of Engineering–Economic
30 Systems, Palo Alto, CA, July 1987.
- 31 [22] YU. G. ZARHIN, Abelian varieties having a reduction of K3 type, *Duke Math J.*
32 **65** no. 3 (1992), 17 pp. [MR 1154181](#). [Zbl 0774.14039](#).
- 33 [23] YU. G. ZARHIN, *Algebra and Cryptography*, Private Communication.
- 34 [24] YU. G. ZARHIN, *On Abel Groups*, Private Communication.
- 35
- 36
- 37
- 38
- 39
- 40
- 41
- 42

(Received: December 24, 2004)

(Revised: April 12, 2006)

1

2

AMS, PROVIDENCE, RHODE ISLAND

3

E-mail: tech-support@ams.org

4

5

GEORGE MASON UNIVERSITY, FAIRFAX, VIRGINIA

6

E-mail: borisv@lk.net

7

<http://borisv.lk.net>

8

THE UNSEEN UNIVERESITY

9

10

11

12

13

14

15

16

17

18

19

20

21

22

23

24

25

26

27

28

29

30

31

32

33

34

35

36

37

38

39

40

41

42