Math 228
2-21-1994

Solutions to Problem Set 14.

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5. Find the general solution of the differential equation $y'' + y = \tan x$, $0 < x < \dfrac{\pi}{2}$.

The homogeneous equation is $(D^2 + 1)y = 0$. $y_1 = \cos x$ and $y_2 = \sin x$ are independent homogeneous solutions.

Look for a solution of the form $y = u_1 \cos x + u_2 \sin x$. Write down the variation of parameters equation and solve for u_1' and u_2':

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \tan x \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} \cos x & -\sin x \\ \sin x & \cos x \end{bmatrix} \begin{bmatrix} 0 \\ \tan x \end{bmatrix} = \begin{bmatrix} -\dfrac{(\sin x)^2}{\cos x} \\ \sin x \end{bmatrix}.$$

Integrate to solve for u_1 and u_2:

$$u_1' = -\frac{(\sin x)^2}{\cos x} = \frac{(\cos x)^2 - 1}{\cos x} = \cos x - \sec x, \quad u_1 = \sin x - \ln|\sec x + \tan x| + c_1,$$

$$u_2' = \sin x, \quad u_2 = -\cos x + c_2.$$

The general solution is

$$y = \sin x \cos x - \cos x \ln|\sec x + \tan x| + c_1 \cos x - \sin x \cos x + c_2 \sin x =$$

$$-\cos x \ln|\sec x + \tan x| + c_1 \cos x + c_2 \sin x. \ \Box$$

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6. Find the general solution of the differential equation $y'' + 9y = 9(\sec 3x)^2$, $0 < x < \dfrac{\pi}{6}$.

The homogeneous equation is $(D^2 + 9)y = 0$. $y_1 = \cos 3x$ and $y_2 = \sin 3x$ are independent homogeneous solutions.

Look for a solution of the form $y = u_1 \cos 3x + u_2 \sin 3x$. Write down the variation of parameters equation and solve for u_1' and u_2':

$$\begin{bmatrix} \cos 3x & \sin 3x \\ -3\sin 3x & 3\cos 3x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 9(\sec 3x)^2 \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 3\cos 3x & -\sin 3x \\ -3\sin 3x & \cos 3x \end{bmatrix} \begin{bmatrix} 0 \\ 9(\sec 3x)^2 \end{bmatrix} = \begin{bmatrix} -3\sec 3x \tan 3x \\ 3\sec 3x \end{bmatrix}.$$

Integrate to solve for u_1 and u_2:

$$u_1' = -3\sec 3x \tan 3x, \quad u_1 = -\sec 3x + c_1,$$

$$u_2' = 3\sec 3x, \quad u_2 = \ln|\sec 3x + \tan 3x| + c_2.$$

The general solution is

$$y = -1 + c_1 \cos 3x + (\sin 3x)\ln|\sec 3x + \tan 3x| + c_2 \sin 3x. \ \Box$$

14. Verify that the functions $y_1 = x$ and $y_2 = xe^x$ satisfy the homogeneous equation corresponding to $x^2y'' - x(x+2)y' + (x+2)y = 2x^3$, $x > 0$; then find a particular solution of the nonhomogeneous equation.

$$y_1 = x, \quad y_1' = 1, \quad y_1'' = 0,$$
$$x^2y_1'' - x(x+2)y_1' + (x+2)y_1 = 0 - x(x+2) + (x+2)x = 0.$$

$y_1 = x$ is a solution.

$$y_2 = xe^x, \quad y_2' = xe^x + e^x, \quad y_2'' = xe^x + 2e^x,$$
$$x^2y_2'' - x(x+2)y_2' + (x+2)y_2 = x^3e^x + 2x^2e^x - x^3e^x - 3x^2e^x - 2xe^x + x^2e^x + 2xe^x = 0.$$

$y_2 = xe^x$ is a solution.

$y_1 = x$ and $y_2 = xe^x$ are *independent* homogeneous solutions, since they are independent as solutions of the equations $(D-1)^2y = 0$.

Look for a solution of the form $y = u_1x + u_2xe^x$. Rewrite the equation as

$$y'' - \frac{x+2}{x}y' + \frac{x+2}{x^2}y = 2x.$$

Write down the variation of parameters equation and solve for u_1' and u_2':

$$\begin{bmatrix} x & xe^x \\ 1 & xe^x + e^x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ 2x \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{x^2e^x} \begin{bmatrix} xe^x + e^x & -xe^x \\ -1 & x \end{bmatrix} \begin{bmatrix} 0 \\ 2x \end{bmatrix} = \frac{1}{x^2e^x} \begin{bmatrix} -2x^2e^x \\ 2x^2 \end{bmatrix} = \begin{bmatrix} -2 \\ 2e^{-x} \end{bmatrix}.$$

Integrate to solve for u_1 and u_2:

$$u_1' = -2, \quad u_1 = -2x + c_1,$$

$$u_2' = 2e^{-x}, \quad u_2 = -2e^{-x} + c_2.$$

The general solution is

$$y = -2x^2 + c_1x - 2x + c_2xe^x = c_1x + c_2xe^x - 2x^2. \square$$

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17. Verify that the functions $y_1 = x^2$ and $y_2 = x^2\ln x$ satisfy the homogeneous equation corresponding to $x^2y'' - 3xy' + 4y = x^2\ln x$, $x > 0$; then find a particular solution of the nonhomogeneous equation.

$$y_1 = x^2, \quad y_1' = 2x, \quad y_1'' = 2,$$
$$x^2y_1'' - 3xy_1' + 4y_1 = 2x^2 - 6x^2 + 4x^2 = 0.$$

$y_1 = x^2$ is a homogeneous solution.

$$y_2 = x^2\ln x, \quad y_2' = 2x\ln x + x, \quad y_2'' = 2\ln x + 3,$$
$$x^2y_2'' - 3xy_2' + 4y_2 = 2x^2\ln x + 3x^2 - 6x^2\ln x - 3x^2 + 4x^2\ln x = 0.$$

$y_2 = x^2\ln x$ is a homogeneous solution.

Since

$$W(y_1, y_2) = \begin{vmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{vmatrix} = x^3 > 0$$

for $x > 0$, the solutions are independent.

Look for a solution of the form $y = u_1 x^2 + u_2 x^2 \ln x$. Rewrite the equation as

$$y'' - \frac{3}{x} y' + \frac{4}{x^2} y = \ln x.$$

Write down the variation of parameters equation and solve for u_1' and u_2':

$$\begin{bmatrix} x^2 & x^2 \ln x \\ 2x & 2x \ln x + x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ \ln x \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{x^3} \begin{bmatrix} 2x \ln x + x & -x^2 \ln x \\ -2x & x^2 \end{bmatrix} \begin{bmatrix} 0 \\ \ln x \end{bmatrix} = \begin{bmatrix} -\frac{1}{x} (\ln x)^2 \\ \frac{1}{x} \ln x \end{bmatrix}.$$

Integrate to solve for u_1 and u_2:

$$u_1' = -\frac{1}{x} (\ln x)^2, \quad u_1 = -\frac{1}{3} (\ln x)^3 + c_1,$$

$$u_2' = \frac{1}{x} \ln x, \quad u_2 = \frac{1}{2} (\ln x)^2 + c_2.$$

The general solution is

$$y = -\frac{x^2}{3} (\ln x)^3 + c_1 x^2 + \frac{x^2}{2} (\ln x)^3 c_2 x^2 \ln x = \frac{x^2}{6} (\ln x)^3 + c_1 x^2 + c_2 x^2 \ln x. \ \square$$

“Growth is wisdom may be exactly measured by decrease in bitterness.” - FRIEDRICH NIETZSCHE