Math 228
3-1-1994

Solutions to Problem Set 17.

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1. Find the inverse Laplace transform of $\dfrac{3}{s^2+4}$.

$$\mathcal{L}^{-1}\left(\frac{3}{s^2+4}\right) = \frac{3}{2}\,\mathcal{L}^{-1}\left(\frac{2}{s^2+4}\right) = \frac{3}{2}\sin 2t.\ \square$$

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2. Find the inverse Laplace transform of $\dfrac{4}{(s-1)^3}$.

$$\mathcal{L}^{-1}\left(\frac{4}{(s-1)^3}\right) = 2\,\mathcal{L}^{-1}\left(\frac{2}{(s-1)^3}\right) = 2t^2e^t.\ \square$$

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6. Find the inverse Laplace transform of $\dfrac{2s-3}{s^2-4}$.

$$\mathcal{L}^{-1}\left(\frac{2s-3}{s^2-4}\right) = 2\,\mathcal{L}^{-1}\left(\frac{s}{s^2-4}\right) - \frac{3}{2}\,\mathcal{L}^{-1}\left(\frac{2}{s^2-4}\right) = 2\cosh 2t - \frac{3}{2}\sinh 2t.\ \square$$

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10. Find the inverse Laplace transform of $\dfrac{2s-3}{s^2+2s+10}$.

$$\mathcal{L}^{-1}\left(\frac{2s-3}{s^2+2s+10}\right) = \mathcal{L}^{-1}\left(\frac{2(s+1)-5}{(s+1)^2+9}\right) =$$

$$2\,\mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+9}\right) - \frac{5}{3}\,\mathcal{L}^{-1}\left(\frac{3}{(s+1)^2+9}\right) = 2e^{-t}\cos 3t - \frac{5}{3}e^{-t}\sin 3t.\ \square$$

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11. Use the Laplace transform to solve the initial value problem $y'' - y' - 6y = 0$, $y(0) = 1$, $y'(0) = -1$.

Transform the original equation:

$$\left(1 - s + s^2Y(s)\right) - (-1 + sY(s)) - 6Y(s) = 0.$$

Simplify and solve for $Y(s)$:

$$Y(s) = \frac{s-2}{s^2-s-6} = \frac{s-2}{(s-3)(s+2)}.$$

Set up the partial fractions decomposition:

$$\frac{s-2}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2},$$

$$s - 2 = A(s+2) + B(s-3).$$

Setting $s=-2$ gives $-4=-5B$, or $B=\frac{4}{5}$. Setting $s=3$ gives $1=5A$, so $A=\frac{1}{5}$. Therefore,

$$Y(s)=\frac{1}{5}\frac{1}{s-3}+\frac{4}{5}\frac{1}{s+2}.$$

The solution is

$$y(t)=\frac{1}{5}e^{3t}+\frac{4}{5}e^{-2t}. \square$$

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15. Use the Laplace transform to solve the initial value problem $y''-2y'-2y=0$, $y(0)=2$, $y'(0)=0$.

Transform the original equation:

$$\left(0-2s+s^2Y(s)\right)-2\left(-2+sY(s)\right)-2Y(s)=0.$$

Simplify and solve for $Y(s)$:

$$Y(s)=\frac{2s-4}{s^2-2s-2}=\frac{2(s-1)-2}{(s-1)^2-3}=2\frac{s-1}{(s-1)^2-3}-\frac{2}{(s-1)^2-3}.$$

The solution is

$$y(t)=2e^t\cosh\sqrt{3}t-\frac{2}{\sqrt{3}}e^t\sinh\sqrt{3}t. \square$$

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17. Use the Laplace transform to solve the initial value problem $y^{(4)}-4y'''+6y''-4y'+y=0$, $y(0)=0$, $y'(0)=1$, $y''(0)=0$, $y'''(0)=1$.

Transform the original equation:

$$\left(-1-0-s^2-0+s^4Y(s)\right)-4\left(0-s-0+s^3Y(s)\right)+6\left(-1-0+s^2Y(s)\right)-4\left(0+sY(s)\right)+Y(s)=0.$$

Simplify and solve for $Y(s)$:

$$Y(s)=\frac{s^2-4s+7}{(s-1)^4}=\frac{(s^2-2s+1)-2s+6}{(s-1)^4}=\frac{1}{(s-1)^2}-\frac{2(s-1)-4}{(s-1)^4}=\frac{1}{(s-1)^2}-\frac{2}{(s-1)^3}+\frac{4}{(s-1)^4}.$$

The solution is

$$y(t)=te^t-t^2e^t+\frac{2}{3}t^3e^t. \square$$

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28. Let

$$f(s)=\int_0^\infty e^{-st}f(t)\,dt.$$

It is possible to show that as long as f satisfies the conditions of Theorem 6.1.2, it is legitimate to differentiate under the integral sign with respect to the parameter s when $s>a$.

(a) Show that $F'(s)=\mathcal{L}\left(-tf(t)\right)$.

$$F'(s)=\frac{d}{ds}\int_0^\infty e^{-st}f(t)\,dt=\int_0^\infty f(t)\left(\frac{d}{ds}e^{-st}\right)\,dt=\int_0^\infty e^{-st}\left(-tf(t)\right)\,dt=\mathcal{L}\left(-tf(t)\right). \square$$

(b) Show that $F^{(n)}(s) = \mathcal{L}\left((-t)^n f(t)\right)$; hence differentiating the Laplace transform corresponds to multiplying the original function by $-t$.

I proved the result for $n = 1$ in part (a). Assume inductively that the result is true for $n - 1$; that is,

$$F^{(n-1)}(s) = \mathcal{L}\left((-t)^{n-1} f(t)\right).$$

Then

$$F^{(n)}(s) = \frac{d}{ds} F^{(n-1)}(s) = \frac{d}{ds} \mathcal{L}\left((-t)^{n-1} f(t)\right) = \frac{d}{ds} \int_0^\infty e^{-st} \left[(-t)^{n-1} f(t)\right]\, dt =$$

$$\int_0^\infty e^{-st}(-t) \left[(-t)^{n-1} f(t)\right]\, dt = \int_0^\infty e^{-st} \left[(-t)^n f(t)\right]\, dt = \mathcal{L}\left((-t)^n f(t)\right).$$

Therefore, the result is true for all $n \ge 1$ by induction. □

29. Use the result of Problem 28 to find the Laplace transform of te^{at}; a is a real number.

$$\mathcal{L}\left(te^{at}\right) = -\frac{d}{ds} \frac{1}{s - a} = \frac{1}{(s - a)^2}. \ \square$$

"Let not thy will roar, when thy power can but whisper." - Thomas Fuller

