Math 228
3-7-1994

Solutions to Problem Set 20.

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2. Find the solution of the initial value problem $y'' + 4y = \delta(t - \pi) - \delta(t - 2\pi)$, $y(0) = 0$, $y'(0) = 0$ by means of the Laplace transform.

Transform both sides:

$$s^2 Y(s) + 4Y(s) = e^{-\pi s} - e^{-2\pi s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s^2+4} e^{-\pi s} - \frac{1}{s^2+4} e^{-2\pi s}.$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \frac{1}{2}\sin 2t.$$

Therefore,

$$y(t) = u_\pi(t)\frac{1}{2}\sin 2(t-\pi) - u_{2\pi}(t)\frac{1}{2}\sin 2(t-2\pi).$$

(The book's answer is the same, since $\sin 2(t-\pi) = \sin 2(t-2\pi) = \sin 2t$.)

It's instructive to draw the graph of the solution. The left side of the original equation is essentially a spring equation. The initial conditions mean that the mass on the spring is released from equilibrium with velocity 0. Obviously, nothing should happen until an external force is applied to the mass. An impulse is applied at $t = \pi$ and an opposite impulse at $t = 2\pi$. Here's what happens.

First, I'll define a step function:

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u[t_, c_] := 0 /; t <= c
u[t_, c_] := 1 /; t > c
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Here is the solution:

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f2[t_] := u[t, N[Pi]] Sin[2(t - Pi)]/2 -
    u[t, N[2Pi]] Sin[2(t - 2Pi)]/2
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Now I can plot the graph:

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Plot[f2[t], {t, 0, 4Pi}]
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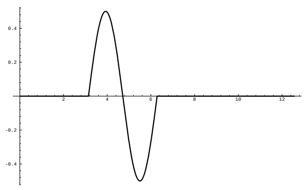


Notice that the displacement is 0 until the first impulse kicks in at $t = \pi$. The mass starts to oscillate, but at $t = 2\pi$ the second impulse occurs and stops the mass in its tracks! □

4. Find the solution of the initial value problem $y'' - y = 2\delta(t-1)$, $y(0) = 1$, $y'(0) = 0$ by means of the Laplace transform.

Transform both sides:

$$\left(-s + s^2 Y(s)\right) - Y(s) = 2e^{-s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{s}{s^2-1} + \frac{2}{s^2-1}e^{-s}.$$

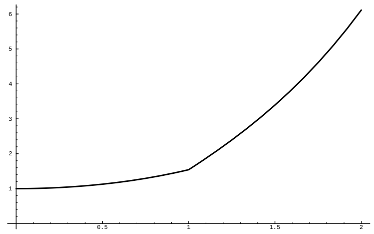
Now

$$\mathcal{L}^{-1}\left(\frac{s}{s^2-1}\right) = \cosh t \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{2}{s^2-1}\right) = 2\sinh t.$$

The term $\dfrac{2}{s^2-1}e^{-s}$ has a shift of $c = 1$, so $\sinh t$ becomes $\sinh(t-1)$. The answer is

$$y(t) = \cosh t + 2u_1(t)\sinh(t-1).$$

Here's a graph of the solution:



The equation represents a system with a forcing term $2\delta(t-1)$. The forcing term acts like a sudden "shove" at $t = 1$. Notice the sharp corner in the graph at $t = 1$ as the system reacts to the "shove". $\square$

7. Find the solution of the initial value problem $y'' + y = \delta(t-\pi)\cos t$, $y(0) = 0$, $y'(0) = 1$ by means of the Laplace transform.

First, observe that

$$\mathcal{L}\left(\delta(t-\pi)\cos t\right) = \cos\pi e^{-\pi s} = -e^{-\pi s}.$$

Transform the differential equation:

$$\left(-1 + 0 + s^2 Y(s)\right) + Y(s) = -e^{-\pi s}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{1}{s^2+1} - e^{-\pi s}\cdot\frac{1}{s^2+1}.$$

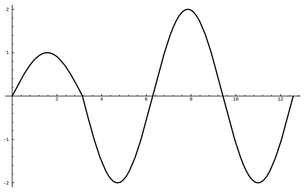
In the second term,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

is shifted by $c = \pi$ to give $u_\pi(t)\sin(t-\pi)$. Hence, the answer is

$$y(t) = \sin t - u_\pi(t)\sin(t-\pi).$$

Here's the graph:



Think of the equation as representing a mass on a spring. The mass begins to oscillate at its natural frequency, and receives an impulse at $t = \pi$ at the *same* frequency (notice that $\cos t$ is a solution to $yy' + y = 0$). The effect is to increase the amplitude of the oscillations without affecting the period. □

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11. Find the solution of the initial value problem $y'' + 2y' + 2y = \cos t + \delta\left(t - \frac{\pi}{2}\right)$, $y(0) = 0$, $y'(0) = 0$ by means of the Laplace transform.

Transform both sides:

$$s^2 Y(s) + 2sY(s) + 2Y(s) = \frac{s}{s^2+1} + e^{-\pi s/2}.$$

Solve for $Y(s)$:

$$Y(s) = \frac{s}{(s^2+1)(s^2+2s+2)} + \frac{1}{(s^2+2s+2)} e^{-\pi s/2}.$$

Apply partial fractions to the first term.

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+2s+2},$$

$$s = (As+B)(s^2+2s+2) + (Cs+D)(s^2+1).$$

Setting $s = 0$ gives $0 = 2B + D$. Save this for future use.
Differentiate the equation to obtain

$$1 = A(s^2+2s+2) + (As+B)(2s+2) + C(s^2+1) + (Cs+D)(2s).$$

Set $s = 0$ again. Then $1 = 2A + 2B + C$. Save this equation, too.
Differentiate again:

$$0 = A(2s+2) + A(2s+2) + (As+B)(2) + 2Cs + 2Cs + (Cs+D)(2).$$

Set $s = 0$ again. Thsi time $0 = 4A + 2B + 2D$, or $0 = 2A + B + D$. Save this equation, too.
Differentiate one more time:

$$0 = 2A + 2A + 2A + 2C + 2C + 2C \quad \text{or} \quad A + C = 0.$$

Here are the 4 saved equations:

$$\begin{array}{ccccccccc} & & 2B & & & + & D & = & 0 \\ 2A & + & 2B & + & C & & & = & 1 \\ 2A & + & B & & & + & D & = & 0 \\ A & & & + & C & & & = & 0 \end{array}$$

Solving simultaneously,

$$A = \frac{1}{5}, \quad B = \frac{2}{5}, \quad C = -\frac{1}{5}, \quad D = -\frac{4}{5}.$$

Therefore,

$$\frac{s}{(s^2+1)(s^2+2s+2)} = \frac{1}{5}\left(\frac{s+2}{s^2+1} - \frac{s+4}{s^2+2s+2}\right).$$

The first term inverts easily:

$$\mathcal{L}^{-1}\left(\frac{1}{5}\frac{s+2}{s^2+1}\right) = \frac{1}{5}\mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) + \frac{1}{5}\mathcal{L}^{-1}\left(\frac{2}{s^2+1}\right) = \frac{1}{5}\cos t + \frac{2}{5}\sin t.$$

The second term contains a quadratic with a middle term. Since

$$s^2+2s+2 = s^2+2s+1+1 = (s+1)^2+1,$$

there is an exponential shift with $c = -1$. Write everything in terms of $s+1$:

$$\frac{1}{5}\frac{s+4}{s^2+2s+2} = \frac{1}{5}\frac{(s+1)+3}{(s+1)^2+1} = \frac{1}{5}\frac{s+1}{(s+1)^2+1} + \frac{1}{5}\frac{3}{(s+1)^2+1}.$$

If $F(s-c) = \dfrac{s+1}{(s+1)^2+1}$, then $c = -1$ and $F(s) = \dfrac{s}{s^2+1}$. This has inverse transform $\cos t$.
If $F(s-c) = \dfrac{3}{(s+1)^2+1}$, then $c = -1$ and $F(s) = \dfrac{3}{s^2+1}$. This has inverse transform $3\sin t$.
Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{5}\frac{s+4}{s^2+2s+2}\right) = \frac{1}{5}e^{-t}\cos t + \frac{3}{5}e^{-t}\sin t.$$

One more term to go! I have to invert $\dfrac{1}{(s^2+2s+2)}e^{-\pi s/2}$. This is a step function shift, so I need to invert $\dfrac{1}{(s^2+2s+2)}$. As above, complete the square and use the exponential shift. First,

$$\frac{1}{(s^2+2s+2)} = \frac{1}{(s+1)^2+1}.$$

Now

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+2s+2)}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}\sin t.$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{1}{(s^2+2s+2)}e^{-\pi s/2}\right) = u_{\pi/2}(t)e^{-(t-\pi/2)}\sin(t-\pi/2).$$

Finally, the answer is

$$y(t) = \frac{1}{5}\cos t + \frac{2}{5}\sin t - \frac{1}{5}e^{-t}\cos t - \frac{3}{5}e^{-t}\sin t + u_{\pi/2}(t)e^{-(t-\pi/2)}\sin(t-\pi/2). \square$$