Solutions to Problem Set 21.

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4. Find the Laplace tranform of $\displaystyle\int_0^t (t-u)^2 \cos 2u \, du$.

$$\mathcal{L}\left(\int_0^t (t-u)^2 \cos 2u \, du\right) = \mathcal{L}(t^2 \star \cos 2t) = \frac{2}{s^3} \cdot \frac{s}{s^2+4} = \frac{2}{s^2(s^2+4)}. \ \square$$

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5. Find the Laplace tranform of $\displaystyle\int_0^t e^{-(t-u)} \sin u \, du$.

$$\mathcal{L}\left(\int_0^t e^{-(t-u)} \sin u \, du\right) = \mathcal{L}(e^{-t} \star \sin t) = \frac{1}{s+1} \cdot \frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)}. \ \square$$

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8. Find the inverse Laplace transform of the function $F(s) = \dfrac{1}{s^4(s^2+1)}$ by using the convolution theorem.

$$\mathcal{L}^{-1}\left(\frac{1}{s^4}\right) = \frac{1}{6}t^3 \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

Therefore,

$$\mathcal{L}\left(\frac{1}{s^4(s^2+1)}\right) = \frac{1}{6}t^3 \star \sin t = \int_0^t \frac{1}{6}(t-u)^3 \sin u \, du =$$

$$\frac{1}{6}\left[-(t-u)^3 \cos u - 3(t-u)^2 \sin u + 6(t-u)\cos u + 6 \sin u\right]_0^t = \sin t + \frac{1}{6}t^3 - t.$$

Here is the work for the integral:

$$\begin{array}{ccccc}
 & \dfrac{d}{du} & & \int du \\
+ & (t-u)^3 & & \sin u \\
 & & \searrow & \\
- & -3(t-u)^2 & & -\cos u \\
 & & \searrow & \\
+ & 6(t-u) & & -\sin u \\
 & & \searrow & \\
- & -6 & & \cos u \\
 & & \searrow & \\
+ & 0 & & \sin u
\end{array}$$

$$\int (t-u)^3 \sin u \, du = -(t-u)^3 \cos u - 3(t-u)^2 \sin u + 6(t-u)\cos u + 6 \sin u + C. \ \square$$

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9. Find the inverse Laplace transform of the function $F(s) = \dfrac{s}{(s+1)(s^2+4)}$ by using the convolution theorem.

$$\mathcal{L}^{-1}\left(\frac{1}{s+1}\right) = e^{-t} \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{s}{s^2+4}\right) = \cos 2t.$$

Therefore,

$$\mathcal{L}\left(\frac{s}{(s+1)(s^2+4)}\right) = e^{-t} \star \cos 2t = \int_0^t e^{-(t-u)} \cos 2u \, du = e^{-t} \int_0^t e^u \cos 2u \, du =$$

$$e^{-t}\left[\frac{2}{5}e^u \sin 2u + \frac{1}{5}e^u \cos 2u\right]_0^t = \frac{1}{5}\cos 2t + \frac{2}{5}\sin 2t - \frac{1}{5}e^{-t}.$$

Here is the work for the integral:

$$\begin{array}{ccc} & \dfrac{d}{du} & \int du \\ + & e^u & \cos 2u \\ - & e^u & \frac{1}{2}\sin 2u \\ + & e^u & -\frac{1}{4}\cos 2u \end{array}$$

$$\int e^u \cos 2u \, du = \frac{1}{2}e^u \sin 2u + \frac{1}{4}e^u \cos 2u - \frac{1}{4}\int e^u \cos 2u \, du,$$

$$\int e^u \cos 2u \, du = \frac{2}{5}e^u \sin 2u + \frac{1}{5}e^u \cos 2u + C. \square$$

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16. Express the solution of the initial value problem $y'' + 4y' + 4y = g(t)$, $y(0) = 2$, $y'(0) = -3$, in terms of a convolution integral.

Transform both sides, letting $\mathcal{L}(g(t)) = G(s)$:

$$\left(3 - 2s + s^2 Y(s)\right) + 4(-2 + sY(s)) + 4Y(s) = G(s).$$

Solve for $Y(s)$:

$$Y(s) = \frac{2s+5}{(s+2)^2} + \frac{G(s)}{(s+2)^2}.$$

Write the first term in terms of $s+2$ and apply the exponential shift:

$$\mathcal{L}^{-1}\left(\frac{2s+5}{(s+2)^2}\right) = \mathcal{L}^{-1}\left(\frac{2(s+2)+1}{(s+2)^2}\right) = 2\,\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s+2)^2}\right) = 2e^{-2t} + te^{-2t}.$$

The second term $\dfrac{G(s)}{(s+2)^2}$ is the product of two transforms. Invert them separately and compute the convolution:

$$\mathcal{L}^{-1}\left(\frac{G(s)}{(s+2)^2}\right) = \int_0^t (t-u)e^{-2(t-u)} g(u) \, du.$$

The answer is

$$y(t) = 2e^{-2t} + te^{-2t} + \int_0^t (t-u)e^{-2(t-u)} g(u) \, du. \square$$

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20. Consider the equation

$$\phi(t) + \int_0^t k(t-\psi)\phi(\psi)d\psi = f(t),$$

in which f and k are known functions, and ϕ is to be determined. Since the unknown function ϕ appears under an integral sign, the given equation is called an **integral equation**; in particular, it belongs to a class of integral equations known as Volterra integral equations. Take the Laplace transform of the given integral equation, and obtain an expression for $\mathcal{L}(\phi(t))$ in terms of the transforms $\mathcal{L}(f(t))$ and $\mathcal{L}(k(t))$ of the given functions f and k. The inverse transform of $\mathcal{L}(\phi(t))$ is the solution of the original integral equation.

Transform the equation and solve for $\Phi(s)$, the transform of ϕ:

$$\Phi(s) + K(s)\Phi(s) = F(s), \quad \Phi(s) = \frac{F(s)}{1+K(s)}. \square$$

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21. Consider the Volterra integral equation

$$\phi(t) + \int_0^t (t-\psi)\phi(\psi)\, d\psi = \sin 2t.$$

(a) Show that if u is a function such that $u''(T) = \phi(t)$, then

$$u''(t) + u(t) - tu'(0) - u(0) = \sin 2t.$$

Set $u'' = \phi$:

$$u'' + \int_0^t (t-w)\phi(w)\, dw = \sin 2t.$$

Integrate by parts:

$$\begin{array}{ccccc} & \dfrac{d}{dw} & & \int dw \\ + & t-w & & \phi(w) \\ & & \searrow & \\ - & -1 & & \int \phi(w)\, dw = u' \\ & & \searrow & \\ + & 0 & & \int u'\, dw = u \end{array}$$

$$\int_0^t (t-w)\phi(w)\, dw = [(t-w)u' + u]_0^t = u - tu'(0) - u(0).$$

Plug this back into the equation above:

$$u'' + u - tu'(0) - u(0) = \sin 2t. \square$$

(b) Show that the given integral equation is equivalent to the initial value problem

$$u''(t) + u(t) = \sin 2t, \qquad u(0) = 0, \quad u'(0) = 0.$$

Set $u(0) = 0$, $u'(0) = 0$ in

$$u'' + u - tu'(0) - u(0) = \sin 2t.$$

Then $u''(t) + u(t) = \sin 2t$. □

(c) Solve the given integral equation by using the Laplace transform.

By Problem 20,

$$\Phi(s) = \frac{\dfrac{2}{s^2+4}}{1 + \dfrac{1}{s^2}} = \frac{2s^2}{(s^2+4)(s^2+1)}.$$

Now

$$\frac{2s^2}{(s^2+4)(s^2+1)} = \frac{2}{3}\left(\frac{4}{s^2+4} - \frac{1}{s^2+1}\right),$$

so

$$\phi(t) = \frac{4}{3}\sin 2t - \frac{2}{3}\sin t. \ \square$$

(d) Solve the initial value problem of part (b) and verify that the solution is the same as that obtained in (c).

The equation is $(D^2+1)u = \sin 2t$. We'll use undetermined coefficients. The homogeneous solution is

$$u_h = c_1\cos t + c_2 \sin t.$$

Guess a particular solution $u_p = A\cos 2t + B\sin 2t$. Then $u_p'' = -4A\cos 2t - 4B\sin 2t$, so

$$u_p'' + u_p = -3A\cos 2t - 3B\sin 2t = \sin 2t.$$

Hence, $A = 0$ and $B = -\dfrac{1}{3}$, and $u_p = -\dfrac{1}{3}\sin 2t$. The general solution is

$$u = c_1\cos t + c_2\sin t - \frac{1}{3}\sin 2t.$$

Plug in the initial conditions. $0 = u(0) = c_1$, so

$$u = c_2 \sin t - \frac{1}{3}\sin 2t.$$

$u' = c_2\cos t - \dfrac{2}{3}\cos 2t$, so $0 = u'(0) = c_2 - \dfrac{2}{3}$ and $c_2 = \dfrac{2}{3}$. Hence,

$$u = \frac{2}{3}\sin t - \frac{1}{3}\sin 2t.$$

Solve for ϕ:

$$\phi = u'' = -\frac{2}{3}\sin t + \frac{4}{3}\sin 2t.$$

This checks with the previous computation. □

"You've always made the mistake of being yourself." - Eugene Ionesco