Math 228
3-24-1994

Solutions to Test 2.

If you want a problem regraded, *do not make any marks on or changes to your exam.* Write an explanation *on a separate piece of paper* and turn it back in with your exam at the end of class. You must ask for any regrades at the time you get your exam eback.

1. Find the general solution to

$$y'' - \frac{x}{x-1}y' + \frac{1}{x-1}y = (x-1)e^{2x},$$

given that $y_1 = x$ and $y_2 = e^x$ are independent homogeneous solutions.

Use variation of parameters, taking the solution to have the form $y = u_1 y_1 + u_2 y_2$. I need to determine u_1 and u_2. Set up the variation of parameters equation:

$$\begin{bmatrix} x & e^x \\ 1 & e^x \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ (x-1)e^{2x} \end{bmatrix},$$

$$\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{(x-1)e^x} \begin{bmatrix} e^x & -e^x \\ -1 & x \end{bmatrix} \begin{bmatrix} 0 \\ (x-1)e^{2x} \end{bmatrix} = \begin{bmatrix} -e^{2x} \\ xe^x \end{bmatrix}.$$

Solve for u_1 and u_2:

$$u_1' = -e^{2x}, \quad u_1 = -\frac{1}{2}e^{2x} + c_1.$$

$$u_2' = xe^x, \quad u_2 = xe^x - e^x + c_2.$$

Here's the work for the integration for u_2:

$$\begin{array}{cccc} & \dfrac{d}{dx} & & \int dx \\ + & x & & e^x \\ & & \searrow & \\ - & 1 & & e^x \\ & & \searrow & \\ + & 0 & \rightarrow & e^x \end{array}$$

$$\int xe^x \, dx = xe^x - e^x + c.$$

The solution is

$$y = x\left(-\frac{1}{2}e^{2x} + c_1\right) + e^x\left(xe^x - e^x + c_2\right) = c_1 x + c_2 e^x + \frac{1}{2}xe^x - e^{2x}. \ \square$$

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2. Find the general solution to

$$y'' + 3y' + 2y = e^x \sin x.$$

I will use undetermined coefficients. Write the equation in the form

$$(D^2 + 3D + 2)y = e^x \sin x.$$

The homogeneous solution is

$$y_h = c_1 e^{-x} + c_2 e^{-2x}.$$

To find a particular solution, complexify:

$$(D^2 + 3D + 2)y = e^{(1+i)x}.$$

$e^{(1+i)x}$ is killed by $D - (1+i)$:

$$[D - (1+i)](D^2 + 3D + 2)y_p = [D - (1+i)]e^{(1+i)x} = 0,$$

$$y_p = Ae^{(1+i)x} + Be^{-x} + Ce^{-2x}.$$

e^{-x} and e^{-2x} are homogeneous solutions: throw them out. Thus,

$$y_p = Ae^{(1+i)x}.$$

Compute the derivatives:

$$y_p' = A(1+i)e^{(1+i)x}, \quad y_p'' = A(1+i)^2 e^{(1+i)x}.$$

Substitute:

$$y_p'' + 3y_p' + 2y_p = \left[(1+i)^2 + 3(1+i) + 2\right] Ae^{(1+i)x} = (5+5i)Ae^{(1+i)x} = e^{(1+i)x}.$$

Thus,

$$A = \frac{1}{5+5i} = \frac{1}{5} \cdot \frac{1}{1+i} \cdot \frac{1-i}{1-i} = \frac{1}{10}(1-i).$$

Hence,

$$y_p = \frac{1}{10}(1-i)e^{(1+i)x} = \frac{1}{10}e^x(1-i)(\cos x + i \sin x) = \frac{1}{10}e^x \left[(\cos x + \sin x) + i(-\cos x + \sin x)\right].$$

The original equation had $e^x \sin x$, which is the imaginary part of $e^{(1+i)x}$, so take the imaginary part of y_p to get a particular solution:

$$\operatorname{im} y_p = \frac{1}{10}e^x(-\cos x + \sin x).$$

The general solution is

$$y = c_1 e^{-x} + c_2 e^{-2x} + \frac{1}{10}e^x(-\cos x + \sin x). \square$$

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3. Find the solution to the initial value problem

$$y'' + y' - 2y = -12(t-3)u_3(t), \qquad y(0) = 0, y'(0) = 0.$$

I'm going to take the transform of both sides. First, I'll do the transform of the right side. Comparing $-12(t-3)u_3(t)$ to $f(t-c)u_c(t)$, I see that $c = 3$, so $f(t-3) = -12(t-3)$. Therefore, $f(t) = -12t$, and

$$F(s) = \mathcal{L}[f(t)] = -\frac{12}{s^2}.$$

Hence,

$$\mathcal{L}\left(-12(t-3)u_3(t)\right) = -\frac{12}{s^2}e^{-3s}.$$

Now transform both sides of the original equation:

$$\left(-0 - 0 \cdot s + s^2 Y(s)\right) + (-0 + sY(s)) - 2Y(s) = -\frac{12}{s^2}e^{-3s}.$$

Solve for $Y(s)$:

$$Y(s) = -\frac{12}{s^2(s+2)(s-1)}e^{-3s}.$$

To invert this, I'll use the step function transform formula. I need to invert $G(s) = -\dfrac{12}{s^2(s+2)(s-1)}$. Use partial fractions:

$$-\frac{12}{s^2(s+2)(s-1)} = \frac{a}{s} + \frac{b}{s^2} + \frac{c}{s+2} + \frac{d}{s-1},$$

$$-12 = as(s+2)(s-1) + b(s+2)(s-1) + cs^2(s-1) + ds^2(s+2).$$

Setting $s = 0$ gives $-12 = -2b$, or $b = 6$. Setting $s = 1$ gives $-12 = 3d$, or $d = -4$. Setting $s = -2$ gives $-12 = -12c$, or $c = 1$. Now the equation reads

$$-12 = as(s+2)(s-1) + 6(s+2)(s-1) + s^2(s-1) - 4s^2(s+2).$$

Set $s = -1$: I get $-12 = 2a - 12 - 2 - 4$, or $a = 3$. Therefore,

$$G(s) = -\frac{12}{s^2(s+2)(s-1)} = \frac{3}{s} + \frac{6}{s^2} + \frac{1}{s+2} - \frac{4}{s-1}.$$

Hence,

$$g(t) = 3 + 6t + e^{-2t} - 4e^t.$$

By the step function transform formula,

$$y(t) = g(t-3)u_3(t).\ \square$$

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4. Find the general solution to

$$2xy'' - y' + \frac{1}{x}y = 0,$$

given that $y_1 = x$ is a (homogeneous) solution.

I'll use reduction of order. Assume the solution has the form $y = vy_1 = vx$. The derivatives are

$$y' = v'x + v, \quad y'' = v''x + 2v'.$$

Substitute:

$$\begin{array}{rcccccl} 2xy'' - y' + \frac{1}{x}y & = & 2x^2v'' & + & 4xv' & & \\ & & & - & xv' & - & v \\ & & & & & + & v \\ & = & 2x^2v'' & + & 3xv' & & = 0. \end{array}$$

Thus,

$$2xv'' + 3v' = 0.$$

Let $u = v'$, so $u' = v''$:

$$2xu' + 3u = 0,$$

$$\int \frac{du}{u} = -\frac{3}{2}\int \frac{dx}{x},$$

$$\ln|u| = -\frac{3}{2}\ln|x| + c,$$

$$u = c_1 x^{-3/2}.$$

Since $u = v'$,

$$v' = c_1 x^{-3/2},$$

$$v = -2c_1 x^{-1/2} + c_2.$$

The solution is

$$y = vx = -2c_1 x^{1/2} + c_2 x. \quad \Box$$

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5. (a) Compute $\mathcal{L}^{-1}\left(\frac{2}{(s-2)(s^2+1)}\right)$.

I'll use the convolution formula. First

$$\mathcal{L}^{-1}\left(\frac{2}{s-2}\right) = 2e^{2t} \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

Hence,

$$\mathcal{L}^{-1}\left(\frac{2}{(s-2)(s^2+1)}\right) = 2e^{2t} \star \sin t = 2\int_0^t e^{2(t-u)} \sin u\, du = 2e^{2t}\int_0^t e^{-2u} \sin u\, du =$$

$$2e^{2t}\left[-\frac{1}{5}e^{-2u}(\cos u + 2\sin u)\right]_0^t = \frac{2}{5}e^{2t} - \frac{2}{5}\cos t - \frac{4}{5}\sin t. \quad \Box$$

(b) Compute $\mathcal{L}^{-1}\left(\frac{s-7}{s^2-2s+10}e^{-2s}\right)$.

I'll use the step function shift formula. First, I need to invert $\frac{s-7}{s^2-2s+10}$:

$$\mathcal{L}^{-1}\left(\frac{s-7}{s^2-2s+10}\right) = \mathcal{L}^{-1}\left(\frac{(s-1)-6}{(s-1)^2+9}\right) =$$

$$\mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2+9}\right) - 6\,\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+9}\right) = e^t\cos 3t - 2e^t \sin 3t.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{s-7}{s^2-2s+10}e^{-2s}\right) = \left(e^{t-2}\cos 3(t-2) - 2e^{t-2}\sin 3(t-2)\right)u_2(t). \quad \Box$$

"No one is so generous as he who has nothing to give." - French proverb