Math 308
2-27-1994

Problem Set 6.

Problem Set 6 is due Monday, March 21, the day after Spring Break.

Reminder: There is a test in class, Monday, March 7. It will cover all the material covered through Wednesday, March 2. There will be a review Sunday, March 6, at 7 p.m., in Yost 104.

These problems on page 295 started out on Problem Set 4, were moved to Problem Set 5, and are now on Problem Set 6. Where will it end?

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1. If V is a vector space over a field F and W is a subspace of V, the **quotient vector space** V/W is the abelian group V/W with scalar multiplication given by

$$a \cdot (v + W) = av + W, \qquad a \in F, v \in V.$$

a. Show that V/W is a vector space over F.

Let $\{w_a\}_{a \in A}$ be a basis for W. Enlarge it to a basis for V; that is, find vectors $\{v_b\}_{b \in B}$ such that $\{w_a\}_{a \in A} \cup \{v_b\}_{b \in B}$ is a basis for V. (This is possible because every linearly independent subset of a vector space may be enlarged to a basis.) In this situation, it's possible to show that $\{v_b + W\}_{b \in B}$ is a basis for V/W.

b. Let $V = \mathbb{R}^3$, considered as an $\mathbb{R}$-vector space in the usual way. Let $W = \langle 1, 1, 1 \rangle$. Use (b) to find a basis for V/W.

c. Let $\langle x^3 \rangle$ be the principal ideal in $\mathbb{R}[x]$ generated by x^3. $\langle x^3 \rangle$ is a subspace of $\mathbb{R}[x]$, if $\mathbb{R}[x]$ is regarded as a vector space over $\mathbb{R}$. In this way, the quotient ring $\dfrac{\mathbb{R}[x]}{\langle x^3 \rangle}$ is also an $\mathbb{R}$-vector space. Find an $\mathbb{R}$-basis for the quotient vector space $\dfrac{\mathbb{R}[x]}{\langle x^3 \rangle}$.

d. Let V_1 and V_2 be subspaces of V. Prove that

$$\frac{V_1}{V_1 \cap V_2} \approx \frac{V_1 + V_2}{V_2}.$$

Conclude that

$$\dim V_1 + \dim V_2 = \dim(V_1 + V_2) + \dim(V_1 \cap V_2).$$

(Use the corresponding result for groups. The only thing you need to verify is that a certain group isomorphism is actually an isomorphism of vector spaces.)

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2. Let $f : V_1 \to V_2$ and $g : V_2 \to V_3$ be linear transformations. Show that the composite $g \cdot f : V_1 \to V_3$ is a linear transformation.

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3. Prove or disprove: If a system of linear equations has more equations than unknowns, then it has no solutions.

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4. Find a basis for $\ker f$ and $\operatorname{im} f$, if f is the linear transformation $\mathbf{R}^4 \to \mathbf{R}^2$ whose matrix relative to the standard bases is given by

$$\begin{bmatrix} 1 & 1 & 0 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

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5. Let $f : V \to W$ be a linear transformation. Suppose f has an inverse $g : W \to V$ (g is the inverse of f as a map of sets.) Show that g must be a linear transformation.

"Genius, by its very intensity, decrees a special path of fire for its vivid power." - Phillips Brooks