11-18-1993

Fourier series.

A Taylor series, such as

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$$

is a sum — an infinite sum — of powers of x. It is useful to represent functions as infinite sums of other kinds of functions. A **Fourier series** is, roughly, a representation of a function as an infinite sum of sines and cosines:

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right)\right).$$

The first thing to notice is that, since sines and cosines are periodic, the sum of sines and cosines will be periodic as well. You can't expect to represent a nonperiodic function such as

$$f(x) = x^2, \qquad -\infty < x < \infty,$$

in this way.

Therefore, I'll consider functions to be defined on an interval $-L \le x < L$ and *continued periodically* to the left and right by $f(x) = f(x + 2L)$.

In addition, I'll assume that at **jump discontinuities** — points $x = c$ where the left- and right-hand limits

$$\lim_{x \to c^-} f(x) \quad \text{and} \quad \lim_{x \to c^+} f(x)$$

aren't equal — the value of the function is the *average value*

$$f(c) = \frac{1}{2}\left(\lim_{x \to c^-} f(x) + \lim_{x \to c^+} f(x)\right).$$

With these assumptions, any **piecewise smooth** function $f(x)$ can be represented as a sum of sines and cosines. The **Fourier coefficients** of $f(x)$ are given by

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right)\,dx \quad \text{and} \quad b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n x}{L}\right)\,dx.$$

The **Fourier series** for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{\pi n x}{L}\right) + b_n \sin\left(\frac{\pi n x}{L}\right)\right).$$

In some cases, it is more efficient to compute the coefficients "all at once" by using the **complex exponential**. Recall that

$$\cos\left(\frac{\pi n x}{L}\right) = \operatorname{re}\exp\left(\frac{\pi n i x}{L}\right) \quad \text{and} \quad \sin\left(\frac{\pi n x}{L}\right) = \operatorname{im}\exp\left(\frac{\pi n i x}{L}\right).$$

Let

$$c_n = \frac{1}{L}\int_{-L}^{L} f(x)\exp\left(\frac{\pi n i x}{L}\right)\,dx.$$

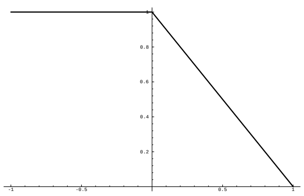
Then

$$a_n = \operatorname{re} c_n \quad \text{and} \quad b_n = \operatorname{im} c_n.$$

The complex exponential formula is less appropriate when the function is symmetric in some way. In fact, if $f(x)$ is either **even** or **odd**, you only need to compute the a_n's or the b_n's, respectively. In these cases, the complex exponential is unnecessary (though you may use it if you wish).

Example. Compute the Fourier expansion for

$$f(x) = \begin{cases} 1 & \text{if } -1 \le x < 0 \\ 1 - x & \text{if } 0 \le x < 1 \end{cases}, \quad \text{and} \quad f(x) = f(x+2) \text{ for all } x.$$



The last line just means the function is defined for all real numbers by "copying" the part from -1 to 1 to the left and right. The function is periodic with period 2, and $L = 1$.

First, compute a_0:

$$a_0 = \frac{1}{1}\int_{-1}^{1} f(x)\,dx = \int_{-1}^{0} 1\,dx + \int_{0}^{1} (1-x)\,dx = [x]_{-1}^{0} + \left[x - \frac{1}{2}x^2\right]_0^1 = 1 + 1 - \frac{1}{2} = \frac{3}{2}.$$

Notice that since f was given in two pieces, I had to split the integral up into two pieces.

Note also that, in general, you can't get a_0 by computing a_n and plugging in $n = 0$. The problem is that the integrals are usually different in kind.

Finally, a_0 is simply the area under $f(x)$ from $-L$ to L (actually, the *signed* area from $-L$ to L). Hence, if you know what the graph of $f(x)$ looks like, it may be easy to find a_0 by looking at the picture.

Next, I'll use the complex exponential to compute the other coefficients:

$$c_n = \frac{1}{1}\int_{-1}^{1} f(x)\exp\left(\frac{\pi n i x}{1}\right)\,dx = \int_{-1}^{0} e^{\pi n i x}\,dx + \int_{0}^{1} (1-x)e^{\pi n i x}\,dx =$$

$$\left[-\frac{i}{\pi n}e^{\pi n i x}\right]_{-1}^{0} + \left[-\frac{i}{\pi n}(1-x)e^{\pi n i x} - \frac{1}{\pi^2 n^2}e^{\pi n i x}\right]_0^1 =$$

$$\left(-\frac{i}{\pi n} + \frac{i}{\pi n}e^{-\pi n i}\right) + \left(0 - \frac{1}{\pi^2 n^2}e^{\pi n i}\right) - \left(-\frac{i}{\pi n} - \frac{1}{\pi^2 n^2}\right) =$$

$$-\frac{1}{\pi^2 n^2}\left(1 - e^{\pi n i}\right) + \frac{i}{\pi n}e^{-\pi n i} = -\frac{1}{\pi^2 n^2}\left(1 - (-1)^n\right) + \frac{i}{\pi n}(-1)^n.$$

Before going on, I want to make some remarks about the computation above. First, whenever an i has appeared in the denominator of a fraction, I've moved it to the top. For example, in doing the integrals you get factors of $\dfrac{1}{\pi n i}$. I've simplified it this way:

$$\frac{1}{\pi n i} = \frac{1}{\pi n i}\cdot\frac{i}{i} = -\frac{i}{\pi n}.$$

(Remember that $i^2 = -1$.) This simplification makes it easier to read off the real and imaginary parts at the end.

The $(-1)^n$ terms came from simplifying the exponentials:

$$e^{\pi n i} = \cos(\pi n) + i \sin(\pi n) = \cos(\pi n) = (-1)^n$$
$$e^{-\pi n i} = \cos(\pi n) - i \sin(\pi n) = \cos(\pi n) = (-1)^n$$

$\sin(\pi n) = 0$ because the sine of an integer multiple of π is 0. $\cos(\pi n) = (-1)^n$ because odd multiples of π have cosine equal to -1, while even multiples of π have cosine equal to 1.

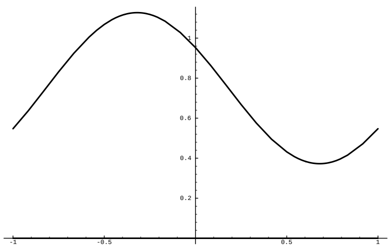
I obtain a_n and b_n by taking the real and imaginary parts of c_n:

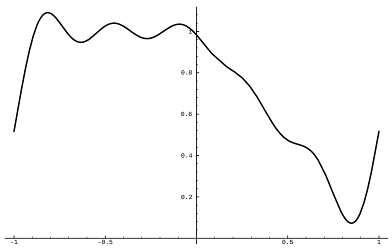
$$a_n = \operatorname{re} c_n = \operatorname{re}\left(-\frac{1}{\pi^2 n^2}\left(1 - (-1)^n\right) + \frac{i}{\pi n}(-1)^n\right) = -\frac{1}{\pi^2 n^2}\left(1 - (-1)^n\right)$$
$$b_n = \operatorname{im} c_n = \operatorname{im}\left(-\frac{1}{\pi^2 n^2}\left(1 - (-1)^n\right) + \frac{i}{\pi n}(-1)^n\right) = \frac{1}{\pi n}(-1)^n$$

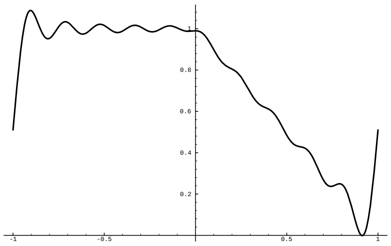
The series is

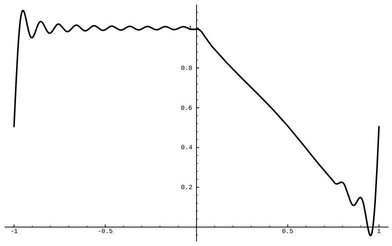
$$f(x) = \frac{3}{4} + \sum_{n=1}^{\infty}\left(\frac{1}{\pi^2 n^2}\left(1 - (-1)^n\right)\cos(\pi n x) + \frac{(-1)^n}{\pi n}\sin(\pi n x)\right).$$

Here is some graphical evidence that the series converges to $f(x)$. The pictures below show the graphs of the sum of the first term, the first 5 terms, the first 10 terms, and the first 20 terms of the Fourier series. Notice how the graphs appear to be converging to a horizontal line on the left and a diagonal line on the right.









Finally, here's the work for the integration by parts from the computation. This is just integration, and has nothing to do with Fourier series *per se*. However, you should work through a few of these yourself to make sure you understand the computations.

$$\begin{array}{ccccc} & \dfrac{d}{dx} & & \int dx \\ + & 1-x & & e^{\pi n i x} \\ & & \searrow & \\ - & -1 & & -\dfrac{i}{\pi n} e^{\pi n i x} \\ & & \searrow & \\ + & 0 & & -\dfrac{1}{\pi^2 n^2} e^{\pi n i x} \end{array}$$

$$\int (1-x) e^{\pi n i x}\,dx = -\frac{i}{\pi n i}(1-x)e^{\pi n i x} - \frac{1}{\pi^2 n^2} e^{\pi n i x} + C.\ \square$$

Example. Compute the Fourier expansion of

$$f(x) = (\sin x)^2, \qquad -\pi \le x < \pi, \quad \text{and} \quad f(x) = f(x + 2\pi) \text{ for all } x.$$

Since $L = \pi$, I need to expand $(\sin x)^2$ in terms

$$\cos x, \sin x, \cos(2x), \sin(2x), \cos(3x), \ldots.$$

Use the double angle formula:

$$(\sin x)^2 = \frac{1}{2}(1 - \cos(2x)) = \frac{1}{2} - \frac{1}{2}\cos(2x).$$

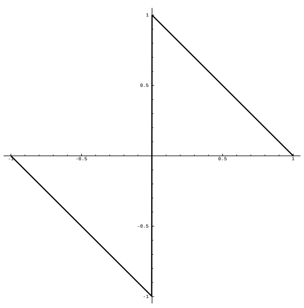
This is an expansion of $(\sin x)^2$ in terms of sines and cosines of the right periods, so it must be the Fourier series! $\square$

A function is **odd** if $f(-x) = -f(x)$ — that is, the graph is symmetric about the origin. A function if **even** if $f(-x) = f(x)$ — that is, the graph is symmetric about the y-axis. Observe that sines are odd and cosines are even. From this, it follows that:

- The Fourier expansion of an even function contains only cosine terms. (This means you only need to compute the a_n's.)
- The Fourier expansion of an odd function contains only sine terms. (This means you only need to compute the b_n's.)

Example. Compute the Fourier expansion of

$$f(x) = \begin{cases} 1 - x & \text{if } 0 \le x < 1 \\ -1 - x & \text{if } -1 \le x < 0 \end{cases}, \quad \text{and} \quad f(x) = f(x+2) \text{ for all } x.$$



$f(x)$ is odd, as you can see by examining its graph. Therefore, I only need to compute the sine coefficients.

$$b_n = \frac{1}{1}\int_{-1}^{1} f(x)\sin(\pi n x)\,dx = \int_{-1}^{0}(-1-x)\sin(\pi n x)\,dx + \int_{0}^{1}(1-x)\sin(\pi n x)\,dx =$$

$$2\int_{0}^{1}(1-x)\sin(\pi n x)\,dx = 2\left[-\frac{1}{\pi n}(1-x)\cos(\pi n x) - \frac{1}{\pi^2 n^2}\sin(\pi n x)\right]_0^1 = \frac{2}{\pi n}.$$

Therefore,

$$f(x) = \sum_{n=1}^{\infty}\frac{2}{\pi n}\sin(\pi n x).\ \square$$

Notice how I used symmetry to combine two integrals into one: If I let $u = -x$, then $du = -dx$, and

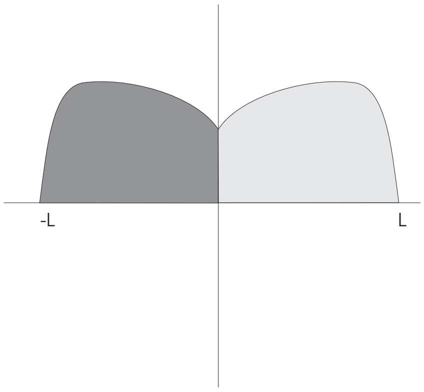
$$\int_{-1}^{0}(-1-x)\sin(\pi n x)\,dx \int_{1}^{0}(-1+u)\sin(-\pi n u)\,(-du) = \int_{0}^{1}(1-u)\sin(\pi n u)\,du = \int_{0}^{1}(1-x)\sin(\pi n x)\,dx.$$

In fact, this will always happen when you compute the coefficients for an odd or even function.

Suppose $f(x)$ is odd. You only need to compute the sine coefficients b_n. Since $\sin\left(\frac{\pi n x}{L}\right)$ is odd, $f(x)\sin\left(\frac{\pi n x}{L}\right)$ is even. By symmetry,

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{\pi n x}{L}\right)\,dx = \frac{2}{L}\int_{0}^{L} f(x)\sin\left(\frac{\pi n x}{L}\right)\,dx.$$

The idea is that if a function is even, it is symmetric about the y-axis. Hence, the integral from $-L$ to L is twice the integral from 0 to L. This is obvious if you think of the integral as giving the area bounded by the graph.



Suppose $f(x)$ is even. You only need to compute the cosine coefficients a_n. Again by symmetry,

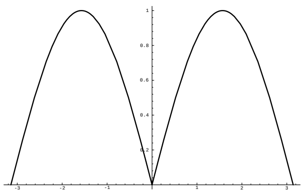
$$a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\left(\frac{\pi n x}{L}\right)\,dx.$$

In fact, you may take these as definitions in case you want to expand a function in *only sines* or *only cosines*. (These kinds of expansions occur, for example, in solving the **heat equation** and the **wave equation**.) In these cases, a function $f(x)$ is given on the interval $0 \le x < L$; the formulas above are used to compute the **Fourier sine series** or **Fourier cosine series** for f.

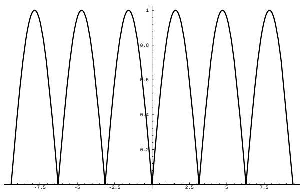
Example. Compute the Fourier cosine series for the even extension of

$$f(x) = \sin x, \qquad 0 \le x < \pi.$$

Here's how you get the even extension. First, take the given function and reflect it about the y-axis to get an even function on $-\pi \le x < \pi$.



Next, copy the graph to the left and right to get a function with period 2π:



This is the even extension of the original function, and this is the function whose Fourier series I'm computing.

First,

$$a_0 = \frac{2}{\pi}\int_0^\pi \sin x\,dx = \frac{4}{\pi}.$$

Next, recall the trig identity

$$\sin a \cos b = \frac{1}{2}\left(\sin(a+b) - \sin(a-b)\right).$$

It follows that

$$a_n = \frac{2}{\pi}\int_0^\pi \sin x \cos(nx)\,dx = \frac{2}{\pi}\int_0^1 \frac{1}{2}\left(\sin(n+1)x - \sin(n-1)x\right)\,dx.$$

Before going on, notice that I only used the part of the function between 0 and *pi* — the original definition. The *expansion* I get will be for the even extension pictured above, but I don't need a formula for the even extension to find the a_n's.

Continue with the computation of the integral. Suppose $n > 1$. Then you can integrate the stuff above as is to obtain

$$a_n = \frac{1}{\pi}\left[-\frac{1}{n+1}\cos(n+1)x + \frac{1}{n-1}\cos(n-1)x\right]_0^\pi =$$

$$\frac{1}{\pi}\left(\frac{1}{n+1}(1-\cos(n+1)\pi)-\frac{1}{n-1}(1-\cos(n-1)\pi)\right).$$

For $n=1$, the integral becomes

$$a_1=\frac{2}{\pi}\int_0^{\pi}\sin x\cos x\,dx=0.$$

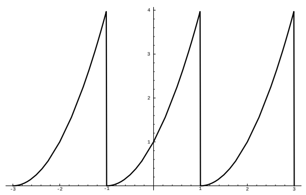
Therefore,

$$f(x)=\frac{2}{\pi}+\sum_{n=1}^{\infty}\frac{1}{\pi}\left(\frac{1}{n+1}(1-\cos(n+1)\pi)-\frac{1}{n-1}(1-\cos(n-1)\pi)\right)\cos kx.\ \square$$

Problems on Fourier series.

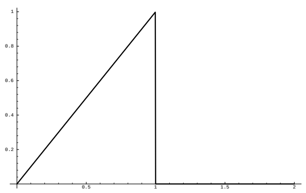
1. Compute the Fourier expansion for

$$f(x)=(x+1)^2,\quad -1\le x<1,\quad \text{and}\quad f(x)=f(x+2)\text{ for all }x.$$



2. Let

$$f(x)=\begin{cases} x & \text{if } 0\le x<1\\ 0 & \text{if } 1\le x<2\end{cases}$$

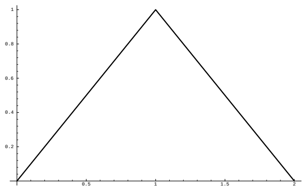


(a) Compute the Fourier sine series for the odd extension of $f(x)$.

(b) Compute the Fourier cosine series for the even extension of $f(x)$.

3. Let

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2 - x & \text{if } 1 \le x < 2 \end{cases}$$



(a) Find the Fourier sine series for the odd extension of f.

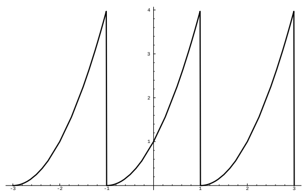
(b) Find the Fourier cosine series for the even extension of f.

(c) Let $h(x)$ denote the even extension of f. What is $h(9)$?

Solutions to the Problems on Fourier series.

1. Compute the Fourier expansion for

$$f(x) = (x+1)^2, \quad -1 \le x < 1, \quad \text{and} \quad f(x) = f(x+2) \text{ for all } x.$$



The function is neither even nor odd, so there may be both sine and cosine terms. I'll use the complex exponential.

First, compute a_0:

$$a_0 = \frac{1}{1}\int_{-1}^{1} (x+1)^2\, dx = \left[\frac{1}{3}(x+1)^3\right]_{-1}^{1} = \frac{8}{3}.$$

Next,

$$c_n = \frac{1}{1}\int_{-1}^{1} (x+1)^2 e^{\pi n i x}\, dx = \left[\left(-\frac{i}{\pi n}(x+1)^2 + \frac{2}{\pi^2 n^2}(x+1) + \frac{2i}{\pi^3 n^3}\right) e^{\pi n i x}\right]_{-1}^{1} =$$

$$\left(-\frac{4i}{\pi n}+\frac{4}{\pi^2 n^2}+\frac{2i}{\pi^3 n^3}\right)e^{\pi n i}-\left(0+0+\frac{2i}{\pi^3 n^3}\right)e^{-\pi n i}=\frac{4}{\pi^2 n^2}(-1)^n-\frac{4i}{\pi n}(-1)^n.$$

Take the real and imaginary parts to obtain a_n and b_n:

$$a_n=\operatorname{re} c_n=\frac{4}{\pi^2 n^2}(-1)^n,\qquad b_n=\operatorname{im} c_n=-\frac{4}{\pi n}(-1)^n.$$

The Fourier expansion is

$$f(x)=\frac{4}{3}+\sum_{n=1}^{\infty}4\cdot(-1)^n\left(\frac{1}{\pi^2 n^2}\cos(\pi n x)-\frac{1}{\pi n}\sin(\pi n x)\right).$$

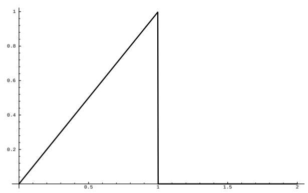
Here's the work for the integration by parts from the computation:

$$\begin{array}{ccccc}
 & \dfrac{d}{dx} & & \int\, dx \\
+ & (x+1)^2 & & e^{\pi n i x} \\
 & & \searrow & \\
- & 2(x+1) & & -\dfrac{i}{\pi n}e^{\pi n i x} \\
 & & \searrow & \\
+ & 2 & & -\dfrac{1}{\pi^2 n^2}e^{\pi n i x} \\
 & & \searrow & \\
- & 0 & & \dfrac{i}{\pi^3 n^3}e^{\pi n i x}
\end{array}$$

$$\int(x+1)^2 e^{\pi n i x}\,dx=\left(-\frac{i}{\pi n}(x+1)^2+\frac{1}{\pi^2 n^2}2(x+1)\frac{2i}{\pi^3 n^3}\right)e^{\pi n i x}+C.\ \square$$

2. Let

$$f(x)=\begin{cases} x & \text{if } 0\le x<1\\ 0 & \text{if } 1\le x<2\end{cases}$$



(a) Compute the Fourier sine series for the odd extension of $f(x)$.

$$b_n=\frac{2}{2}\int_0^2 f(x)\sin\left(\frac{\pi n x}{2}\right)\,dx=\int_0^1 x\sin\left(\frac{\pi n x}{2}\right)\,dx+0=$$

$$\left[-\frac{2}{\pi n}x\cos\left(\frac{\pi n x}{2}\right)+\frac{4}{\pi^2 n^2}\sin\left(\frac{\pi n x}{2}\right)\right]_0^1=-\frac{2}{\pi n}\cos\left(\frac{\pi n}{2}\right)+\frac{4}{\pi^2 n^2}\sin\left(\frac{\pi n}{2}\right).$$

The Fourier sine series for $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} \left(-\frac{2}{\pi n} \cos\left(\frac{\pi n x}{2}\right) + \frac{4}{\pi^2 n^2} \sin\left(\frac{\pi n x}{2}\right) \right) \sin\left(\frac{\pi n x}{2}\right). \square$$

(b) Compute the Fourier cosine series for the even extension of $f(x)$.

First,

$$a_0 = \frac{2}{2} \int_0^2 f(x)\, dx = \int_0^1 x\, dx = \frac{1}{2}.$$

Next,

$$a_n = \frac{2}{2} \int_0^2 f(x) \cos\left(\frac{\pi n x}{2}\right)\, dx = \int_0^1 x \cos\left(\frac{\pi n x}{2}\right)\, dx + 0 =$$

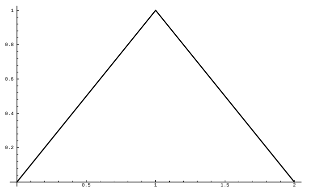
$$\left[\frac{2}{\pi n} x \sin\left(\frac{\pi n x}{2}\right) + \frac{4}{\pi^2 n^2} \cos\left(\frac{\pi n x}{2}\right) \right]_0^1 = \frac{2}{\pi n} \sin\left(\frac{\pi n}{2}\right) + \frac{4}{\pi^2 n^2} \cos\left(\frac{\pi n}{2}\right) - \frac{4}{\pi^2 n^2}.$$

The Fourier cosine series for $f(x)$ is

$$f(x) = \frac{1}{4} + \sum_{n=1}^{\infty} \left(\frac{2}{\pi n} \sin\left(\frac{\pi n x}{2}\right) + \frac{4}{\pi^2 n^2} \cos\left(\frac{\pi n x}{2}\right) - \frac{4}{\pi^2 n^2} \right) \cos\left(\frac{\pi n x}{2}\right). \square$$

3. Let

$$f(x) = \begin{cases} x & \text{if } 0 \le x < 1 \\ 2 - x & \text{if } 1 \le x < 2 \end{cases}$$



(a) Find the Fourier sine series for the odd extension of f.

$$b_n = \frac{2}{2} \int_0^2 f(x) \sin\frac{\pi n x}{2}\, dx = \int_0^1 x \sin\frac{\pi n x}{2}\, dx + \int_1^2 (2 - x) \sin\frac{\pi n x}{2}\, dx = \frac{8}{\pi^2 n^2} \sin\frac{\pi n}{2}.$$

The Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} \frac{8}{\pi^2 n^2} \sin\frac{\pi n}{2} \sin\frac{\pi n x}{2}. \square$$

(b) Find the Fourier cosine series for the even extension of f.

First,

$$a_0 = \frac{2}{2}\int_0^2 f(x)\,dx = 1.$$

Next,

$$a_n = \frac{2}{2}\int_0^2 f(x)\cos\frac{\pi n x}{2}\,dx = \int_0^1 x\cos\frac{\pi n x}{2}\,dx + \int_1^2 (2-x)\cos\frac{\pi n x}{2}\,dx = \frac{4}{\pi^2 n^2}\left(2\cos\frac{\pi n}{2} - \cos\pi n - 1\right).$$

The Fourier cosine series is

$$f(x) = \frac{1}{2} + \sum_{n=1}^{\infty} \frac{4}{\pi^2 n^2}\left(2\cos\frac{\pi n}{2} - \cos\pi n - 1\right)\cos\frac{\pi n x}{2}. \quad \square$$

(c) Let $h(x)$ denote the even extension of f. What is $h(9)$?

The even extension is

$$h(x) = \begin{cases} x+2 & \text{if } -2 \le x < -1 \\ -x & \text{if } -1 \le x < 0 \\ x & \text{if } 0 \le x < 1 \\ 2-x & \text{if } 1 \le x < 2 \end{cases}, \quad \text{and} \quad h(x) = h(x+4) \text{ for all } x.$$

(The best way to see this is to take the picture for f, reflect the graph across the y-axis, then figure out what the equations of the reflected pieces have to be.)

Then $h(9) = h(5) = h(1) = 2 - 1 = 1$. $\square$