Math 227
9-19-1993

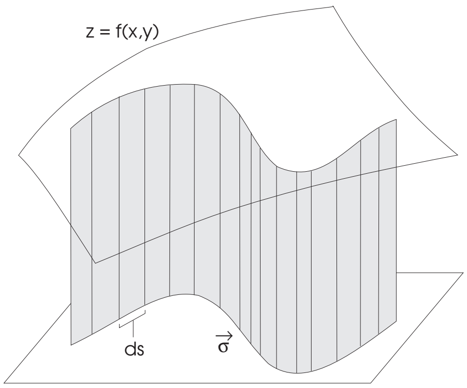
Path integrals.

If f is a function and $\vec{\sigma}$ is a path, the **path integral** of f over $\vec{\sigma}$ is denoted

$$\int_{\vec{\sigma}} f\,ds.$$

It is defined as the limit of a sum of rectangle areas, where the bases of the rectangles lie along the path and the heights of the rectangles are determined by f.

The picture below depicts a rectangle sum approximation to a path integral $\displaystyle\int_{\vec{\sigma}} f\,ds$. I have drawn the picture for a curve in the x-y-plane, but this discussion applies, with the obvious changes, to paths in higher dimensions.



A small piece of the curve (i.e. the base of a typical rectangle) has length ds, where $ds = \sqrt{dx^2 + dy^2}$. The height of the rectangle is $f(x, y)$.

Represent the curve in parametrized form: $\vec{\sigma}(t) = \langle x(t), y(t)\rangle$, or $x = x(t)$, $y = y(t)$. Then operating formally with the differentials,

$$\sqrt{dx^2 + dy^2} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Plug $x = x(t)$, $y = y(t)$ into $f(x, y)$ to express f in terms of t. The area of a little rectangle is

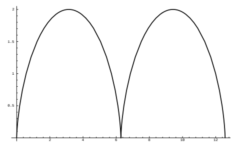
$$f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Finally, if $a \le t \le b$ for the path $\vec{\sigma}$, then

$$\int_{\vec{\sigma}} f\,ds = \int_a^b f(x(t), y(t)) \cdot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\,dt.$$

Example. Compute $\int_{\vec{\sigma}} \sqrt{2y}\,ds$, where $\vec{\sigma}$ is the first arc of the cycloid

$$x = t - \sin t, \quad y = 1 - \cos t.$$



The first arc of the cycloid corresponds to the parameter interval $0 \le t \le 2\pi$.
The velocity is

$$\vec{\sigma}\,'(t) = \langle 1 - \cos t, \sin t \rangle, \text{ so } |\vec{\sigma}\,'(t) = \sqrt{(1 - \cos t)^2 + (\sin t)^2} = \sqrt{2 - 2\cos t}.$$

Hence, $ds = \sqrt{2 - 2\cos t}\, dt$.
The integrand is

$$\sqrt{2y} = \sqrt{2(1 - \cos t)}.$$

Therefore,

$$\int_{\vec{\sigma}} \sqrt{2y}\, ds = \int_0^{2\pi} \sqrt{2(1 - \cos t)}\sqrt{2 - 2\cos t}\, dt = \int_0^{2\pi} 2(1 - \cos t)\, dt = 2\left[t - \sin t\right]_0^{2\pi} = 4\pi \approx 12.56637. \quad \square$$

Line integrals.

If $\vec{F}$ is a vector field and $\vec{\sigma}$ is a path,

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}$$

is the **line integral** of $\vec{F}$ along $\vec{\sigma}$.

You can compute a line integral by parametrizing the path. If $\vec{\sigma} = \langle x(t), y(t) \rangle$, then

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_a^b \vec{F}(x(t), y(t)) \cdot \vec{\sigma}'(t)\, dt.$$

Physical interpretation: If $\vec{F}$ is a force field and you think of $\vec{\sigma}$ as traced out by a moving particle, then the line integral gives the work the particle does in moving through the field.

The discussion above pertains to curves and fields in the plane. Everything extends in the obvious way to higher dimensions.

Example. Compute $\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}$, where $\vec{F} = \langle x, y, x + y - 1 \rangle$ and $\vec{\sigma}$ is the segment from $(1, 1, 1)$ to $(2, 3, 4)$.

If P and Q are points, the segment from P to Q is parametrized by

$$(1 - t) \cdot P + t \cdot Q, \quad 0 \le t \le 1.$$

(You can derive this formula by observing that $Q - P$ is a vector parallel to the line $\overrightarrow{PQ}$, then using this vector and the point P to write down parametric equations for the line.)

Using this formula, I'll parametrize the segment by

$$\vec{\sigma} = (1 + t, 1 + 2t, 1 + 3t), \quad 0 \le t \le 1.$$

(In fact, it doesn't matter how I parametrize the segment.)
Now

$$\vec{\sigma}'(t) = \langle 1, 2, 3\rangle \text{ and } \vec{F}(t) = 1 + t, 1 + 2t, 1 + 3t\rangle, \text{ so } \vec{F}(t) \cdot \vec{\sigma}'(t) = 6 + 14t.$$

Therefore,

$$\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_0^1 (6 + 14t)\, dt = \left[6t + 7t^2\right]_0^1 = 13.\ \square$$

Reparametrization.

In the last example, I computed the line integral along a segment connecting two points. What would have happened if I had used a different parametrization for the segment?

I'll look at the general question: What happens to the line integral along a path if you **reparametrize** the path (i.e. if you parametrize the path in a different way)?

If $\vec{\sigma}$ for $a \le t \le b$ is a path and $h : [c, d] \to [a, b]$ is a *one-to-one* function, then

$$\vec{\tau}(s) = \vec{\sigma}(h(s))$$

is a **reparametrization** of $\vec{\sigma}$. This is a fancy way of saying that $\vec{\tau}$ "covers the same ground" as $\vec{\sigma}$, but $\vec{\tau}$ may do so at a different rate, and possibly in a different direction.

For example,

$$\vec{\sigma}(t) = \langle \cos t, \sin t\rangle, \quad 0 \le t \le 2\pi, \quad \text{and} \quad \vec{\tau}(s) = \langle \cos -2s, \sin -2s, \quad 0 \le s \le \pi$$

are parametrizations of the unit circle. The first parametrization traces out the circle counterclockwise, with constant speed 1. The second parametrization traces out the circle clockwise, with constant speed 2. Either one is a reparametrization of the other.

If paths $\vec{\sigma}$ and $\vec{\tau}$ are reparametrizations of each other, they have the **same orientation** if they start at the same point and end at the same point. The only other possibility is that they cover the same ground in opposite directions, in which case they have **opposite orientations**.

Here is how path integrals and line integrals behave under reparametrization. Suppose that $\vec{\sigma}$ and $\vec{\tau}$ are reparametrizations of a given curve.

1. If $\vec{\sigma}$ and $\vec{\tau}$ have the *same* orientation, then

$$\int_{\vec{\sigma}} f\, ds = \int_{\vec{\tau}} f\, ds \quad \text{and} \quad \int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = \int_{\vec{\tau}} \vec{F} \cdot \overrightarrow{ds}.$$

2. If $\vec{\sigma}$ and $\vec{\tau}$ have the *opposite* orientation, then

$$\int_{\vec{\sigma}} f\, ds = \int_{\vec{\tau}} f\, ds \quad \text{and} \quad \int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds} = -\int_{\vec{\tau}} \vec{F} \cdot \overrightarrow{ds}.$$

In other words, path integrals are not affected by reparametrization, even when the orientation changes. The absolute value of a line integral doesn't change under reparametrization, but the sign "flips" if the two paths go in opposite directions. This is what happens when you swap the limits on a single-variable integral:

$$\int_1^5 x^2\, dx = -\int_5^1 x^2\, dx.$$

Path independence of gradient fields.

The line integral of a gradient field

$$\int_{\vec{\sigma}} \nabla f \cdot \overrightarrow{ds}$$

is **path independent** in the sense that the value of the integral depends only on the endpoints of the path. In particular, if $\vec{\sigma}$ is a path from P to Q, then

$$\int_{\vec{\sigma}} \nabla f \cdot \overrightarrow{ds} = f(Q) - f(P).$$

This is, of course, how we use the Fundamental Theorem of Calculus to compute integrals:

$$\int_1^3 \frac{dx^2}{dx}\, dx = \left[x^2\right]_1^3 = 9 - 1 = 8.$$

It is easy to prove path independence using Green's theorem (in the plane) or Stokes' theorem (in higher dimensions).

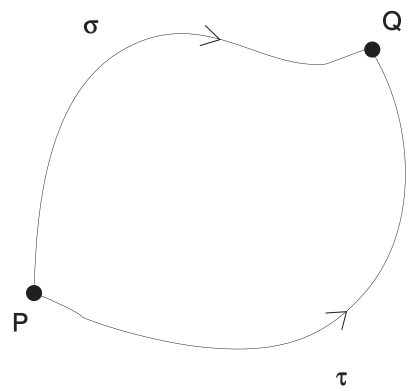
Here are two easy consequences of path independence:

1. If $\vec{\sigma}$ and $\vec{\tau}$ are paths from P to Q, then

$$\int_{\vec{\sigma}} \nabla f \cdot \overrightarrow{ds} = \int_{\vec{\tau}} \nabla f \cdot \overrightarrow{ds}.$$

2. If $\vec{\sigma}$ is a **closed path** (that is, $P = Q$), then

$$\int_{\vec{\sigma}} \nabla f \cdot \overrightarrow{ds} = 0.$$



Note: These properties hold *only* for line integrals of *gradient fields.*

Example. Compute

$$\int_{\vec{\sigma}} (2xy + 2z)\, dx + x^2\, dy + (2x - 3z^2)\, dz,$$

where

$$\vec{\sigma}(t) = \langle t^4 - t, \sin \frac{\pi}{2} t, \tan \frac{\pi}{4} t\rangle, \quad 0 \le t \le 1.$$

I'm using **differential notation** for the line integral. It is the same as $\displaystyle\int_{\vec{\sigma}} \vec{F} \cdot \overrightarrow{ds}$, where

$$\vec{F} = \langle 2xy + 2z, x^2, 2x - 3z^2\rangle.$$

To understand the connection, take the dot product of $\vec{F}$ with the formal vector

$$\langle\, dx,\, dy,\, dz\rangle = \left\langle \frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}\right\rangle dt = \vec{\sigma}\,'(t)\, dt = \overrightarrow{ds}.$$

This gives the integrand above.

To compute the integral directly, I'd substitute

$$x = t^4 - t, \quad y = \sin\frac{\pi}{2}t, \quad z = \tan\frac{\pi}{4}t$$

in the integrand, replace dx, dy, and dz by $\dfrac{dx}{dt}$, $\dfrac{dy}{dt}$, and $\dfrac{dz}{dt}$, respectively, and then integrate with respect to t.

That sounds pretty complicated! Is there an easier way?

I'll show that the integrand is a gradient field. This means that I need a function f such that

$$\frac{\partial f}{\partial x} = 2xy + 2z, \quad \frac{\partial f}{\partial y} = x^2, \quad \frac{\partial f}{\partial z} = 2x - 3z^2.$$

Recover f by doing a series of "partial integrations". First, integrate $\dfrac{\partial f}{\partial x}$ with respect to x:

$$\frac{\partial f}{\partial x} = 2xy + 2z, \text{ so } f = \int (2xy + 2z)\, dx = x^2 y + 2xz + C(y, z).$$

(Since the integral is with respect to x, y and z are constant, and could appear in the arbitrary constant — hence $C(y, z)$.)

Differentiate f with respect to y and set the result equal to $\dfrac{\partial f}{\partial y}$:

$$x^2 = \frac{\partial f}{\partial y} = x^2 + \frac{\partial C}{\partial y}, \text{ so } \frac{\partial C}{\partial y} = 0.$$

This says C is constant with respect to y, so it doesn't involve y — it's at most a function of z. Write $C(y, z) = K(z)$. Then

$$f = x^2 y + 2xz + K(z).$$

Differentiate f with respect to z and set the result equal to $\dfrac{\partial f}{\partial z}$:

$$2x - 3z^2 = \frac{\partial f}{\partial z} = 2x + \frac{dK}{dz}, \text{ so } -3z^2 = \frac{dK}{dz}.$$

I get $K = -z^3$. Any arbitrary constant will be *numerical*, but the derivative of a number is 0. Hence, I just write $-z^3$ instead of $-z^3 + D$, since I just want *some* function f that works.

Anyway, if I plug K back into f, I get

$$f = x^2 y + 2xz - z^3.$$

The integrand is indeed a gradient field. I can compute the integral by plugging in the endpoints. The endpoints of the path are

$$\vec{\sigma}(0) = \langle 0, 0, 0\rangle \text{ and } \vec{\sigma}(1) = \langle 0, 1, 1\rangle.$$

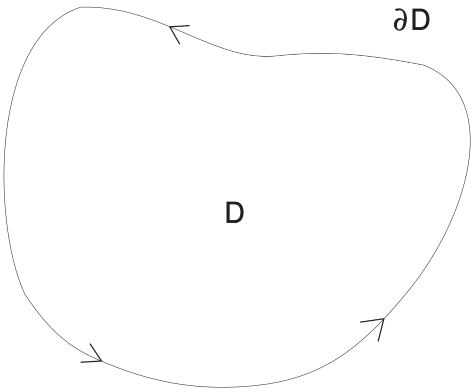
Therefore,

$$\int_{\vec{\sigma}} (2xy + 2z)\, dx + x^2\, dy + (2x - 3z^2)\, dz = f(0, 1, 1) - f(0, 0, 0) = -1 - 0 = -1. \;\Box$$

Green's theorem.

Green's theorem is one of the major results of vector integral calculus. It relates a double integral over a region D in the plane to a line integral around the boundary ∂D:

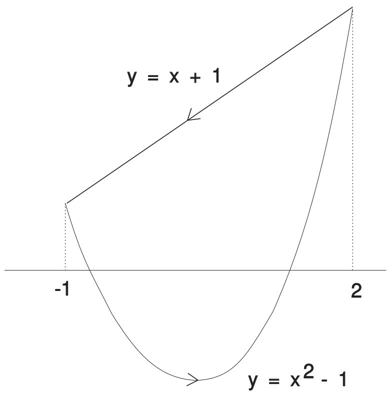
$$\int_{\partial D} P\,dx + Q\,dy = \int\int_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dx\,dy.$$



The boundary must be traversed in the *positive* direction. Imagine walking along the boundary. If you extend your right arm, it should be above the *outside* of the region.

Example. C consists of the part of the parabola $y = x^2 - 1$ from $(-1, 0)$ to $(2, 3)$, followed by the segment from $(2, 3)$ to $(-1, 0)$. Compute

$$\int_C 3y^2\,dx + (6x + 6xy)\,dy.$$



To compute the line integral directly, I'd need to parametrize the arc of the parabola and the segment, compute the integral along each piece, and add the results. This sounds like a lot of work, so I'll use Green's theorem instead.

The segment is part of the line $y = x + 1$. The region is described by the inequalities

$$-1 \le x \le 2, \quad x^2 - 1 \le y \le x + 1.$$

$P = 3y^2$ and $Q = 6x + 6xy$, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 6 + 6y - 6y = 6.$$

By Green's theorem,

$$\int_C 3y^2\,dx + (6x + 6xy)\,dy = \int_{-1}^{2}\int_{x^2-1}^{x+1} 6\,dy\,dx = 6\int_{-1}^{2}\left(x + 1 - (x^2 - 1)\right)\,dx = 27.\ \square$$

In the last example, I used Green's theorem to convert a line integral to a double integral. You can also go the other way: Convert a double integral to a line integral. As a particular example of this, here is how to use Green's theorem to compute the area of a region D.

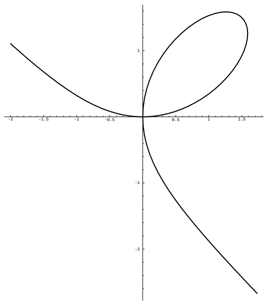
Notice that

$$\int_{\partial D} x\,dy = \int\int_D (1 - 0)\,dx\,dy = \int\int_D dx\,dy.$$

The last double integral represents the area of D. Therefore, the line integral $\displaystyle\int_{\partial D} x\,dy$ gives the area of D.

Example. Use Green's theorem to compute the area of the loop of the **Folium of Descartes**

$$x^3 + y^3 = 3xy.$$



I'll use a little trick to parametrize the curve. Set $y = xt$. Then

$$x^3 + x^3t^3 = 3x^2t, \quad x^3(1 + t^3) = 3x^2t, \quad x = \frac{3t}{1 + t^3}.$$

From $y = xt$, I get $y = \dfrac{3t^2}{1 + t^3}$.

You can check by plugging in points that the loop is traced out as t goes from 0 to $+\infty$.

$$\frac{dy}{dt} = \frac{3t(2 - t^3)}{(1 + t^3)^2} \quad \text{and} \quad x = \frac{3t}{1 + t^3} \quad \text{so} \quad x\frac{dy}{dt} = \frac{3t}{1 + t^3}\frac{3t(2 - t^3)}{(1 + t^3)^2}.$$

Therefore, the area is

$$\int_C x\,dy = \int_0^{+\infty} \frac{3t}{1 + t^3}\frac{3t(2 - t^3)}{(1 + t^3)^2}\,dt = 3\int_0^{+\infty} \frac{2 - t^3}{(1 + t^3)^3}\,3t^2\,dt.$$

Let $u = 1 + t^3$, so $du = 3t^2\,dt$. When $t = 0$, $u = 1$, and as $t \to +\infty$, $u \to +\infty$. So the integral becomes

$$3\int_1^{+\infty} \frac{3 - u}{u^3}\,du = 3\int_1^{+\infty}\left(\frac{3}{u^3} - \frac{1}{u^2}\right)\,du = 3\left[-\frac{3}{2u^2} + \frac{1}{u}\right]_1^{+\infty} = \frac{3}{2}.\ \square$$

Problems on Path integrals, Line integrals, and Green's theorem.

1. Compute $\int_{\vec{\sigma}} x\sqrt{x^2-y^2}\,ds$, where $\vec{\sigma}$ is the loop

$$(x^2+y^2)^2 = x^2-y^2, \quad x \geq 0.$$

(Hint: To paraemtrize the curve, try $y = xt$.)

2. Suppose $\vec{\sigma}$ is a curve that does not pass through the origin, starts at $(3,4)$, and ends at $(5,12)$. Compute

$$\int_{\vec{\sigma}} \frac{x}{x^2+y^2}\,dx + \frac{y}{x^2+y^2}\,dy.$$

3. Let $\vec{\sigma}$ be the curve of intersection of the parabolic cylinder $y = 1-x^2$, $-1 \leq x \leq 1$, and the plane $z = y$, traversed in the direction of increasing x. Compute the work done by a particle moving along this path through the force field $\vec{F} = \langle z, -x, -x\rangle$.

4. Find the area inside the loop of the **Bernoulli lemniscate**

$$(x^2+y^2)^2 = 2(x^2-y^2).$$

(Hint: To parametrize the curve, try $y = xt$.)

5. For what closed curve C is

$$\int_C y^3\,dx + (3x-x^3)\,dy$$

largest? What is the largest value?

6. Compute

$$\int_C (xy^2z^2 + x^2y^3)\,dx + (x^2yz^2 + x^3y^2)\,dy + x^2y^2z\,dz,$$

where C is a path from $(1,1,1)$ to $(0,0,0)$.

7. A particle moves through the force field $\vec{F} = \langle x^2-y^2, 2xy\rangle$.

(a) Compute the work done by the particle in moving from $(0,0)$ to $(2,4)$ along the path $\vec{\sigma}(t) = \langle t, t^2\rangle$, $0 \leq t \leq 2$.

(b) Compute the work done by the particle in moving from $(0,0)$ to $(2,4)$ along the path $\vec{\phi}(t) = \langle t, 2t\rangle$, $0 \leq t \leq 2$.

(c) Why do you get different results in moving from $(0,0)$ to $(2,4)$ along different paths?

8. Compute the work done by the particle moving through the field

$$\vec{F} = \langle 4x^3y - xy^2 + \frac{1}{x^2+1}, 6x + x^4 - x^2y + y^{10}\rangle$$

counterclockwise around the unit circle $\vec{\sigma}(t) = \langle \cos t, \sin t\rangle$, $0 \leq t \leq 2\pi$.

Solutions to the Problems on Path integrals, Line integrals, and Green's theorem.

1. Compute $\int_{\vec{\sigma}} x\sqrt{x^2-y^2}\,ds$, where $\vec{\sigma}$ is the loop

$$(x^2+y^2)^2 = x^2-y^2, \quad x \geq 0.$$

(Hint: To paraemtrize the curve, try $y = xt$.)

Following the hint, set $y = xt$. Then

$$(x^2 + x^2t^2)^2 = x^2 - x^2t^2, \quad x^4(1+t^2)^2 = x^2(1-t^2), \quad x^2 = \frac{1-t^2}{(1+t^2)^2}, \quad x = \frac{\sqrt{1-t^2}}{1+t^2}.$$

Since $y = xt$,

$$y = \frac{t\sqrt{1-t^2}}{1+t^2}.$$

The loop is traced out from $t = -1$ to $t = 1$ (since for $t = \pm 1$, $(x, y) = (0, 0)$).

I need to compute $|\vec{\sigma}'(t)|$, but the algebra is rather ugly — so I'll omit some of the worst computations.

$$\frac{dx}{dt} = \frac{t(t^2-3)}{\sqrt{1-t^2}(1+t^2)^2}, \quad \frac{dy}{dt} = -\frac{3t^2-1}{\sqrt{1-t^2}(1+t^2)^2},$$

$$\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 = \frac{1}{(1-t^2)(1+t^2)}, \quad \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} = \frac{1}{\sqrt{(1-t^2)(1+t^2)}}.$$

The integrand is

$$x\sqrt{x^2 - y^2} = \frac{\sqrt{1-t^2}}{1+t^2} \cdot \frac{1-t^2}{1+t^2}.$$

Hence,

$$\int_{\vec{\sigma}} x\sqrt{x^2-y^2}\, ds = \int_{-1}^{1} \frac{1-t^2}{(1+t^2)^{5/2}}\, dt = \frac{2\sqrt{2}}{3} \approx 0.94281. \square$$

2. Suppose $\vec{\sigma}$ is a curve that does not pass through the origin, starts at $(3, 4)$, and ends at $(5, 12)$. Compute

$$\int_{\vec{\sigma}} \frac{x}{x^2+y^2}\, dx + \frac{y}{x^2+y^2}\, dy.$$

If $f(x, y) = \dfrac{1}{2}\ln(x^2 + y^2)$, then

$$\nabla f = \langle \frac{x}{x^2+y^2}, \frac{y}{x^2+y^2} \rangle.$$

Hence, the field is a gradient field, and the line integral is path independent. To compute the integral, plug the endpoints into f:

$$\int_{\vec{\sigma}} \frac{x}{x^2+y^2}\, dx + \frac{y}{x^2+y^2}\, dy = f(5, 12) - f(3, 4) = \frac{1}{2}(\ln 169 - \ln 25) \approx 0.95551. \square$$

3. Let $\vec{\sigma}$ be the curve of intersection of the parabolic cylinder $y = 1 - x^2$, $-1 \le x \le 1$, and the plane $z = y$, traversed in the direction of increasing x. Compute the work done by a particle moving along this path through the force field $\vec{F} = \langle z, -x, -x \rangle$.

Project the curve into the x-y-plane. The projection may be parametrized by $x = t$, $y = 1 - t^2$, for $-1 \le t \le 1$. Since $z = y$, the curve may be parametrized by

$$x = t, \quad y = 1 - t^2, \quad z = 1 - t^2, \qquad -1 \le t \le 1.$$

Next,

$$\vec{\sigma}'(t) = \langle 1, -2t, -2t \rangle, \quad \vec{F}(t) = \langle 1 - t^2, -t, -t \rangle, \text{ so } \vec{F} \cdot \vec{\sigma} = 1 + 3t^2.$$

The work done is

$$W = \int_{-1}^{1} (1 + 3t^2)\, dt = \left[t + t^3\right]_{-1}^{1} = 4. \ \square$$

4. Find the area inside the loop of the **Bernoulli lemniscate**

$$(x^2 + y^2)^2 = 2(x^2 - y^2).$$

(Hint: To parametrize the curve, try $y = xt$.)

Let $y = xt$. Then

$$(x^2 + x^2t^2)^2 = 2(x^2 - x^2t^2), \quad x^4(1 + t^2)^2 = 2x^2(1 - t^2), \quad x = \frac{\sqrt{2}\sqrt{1 - t^2}}{1 + t^2}.$$

Since $y = xt$,

$$y = \frac{\sqrt{2}t\sqrt{1 - t^2}}{1 + t^2}.$$

The loop is traced out as t goes from -1 to 1.
I'll use Green's theorem to compute the area.

$$\frac{dy}{dt} = \sqrt{2}\frac{1 - 3t^2}{(1 + t^2)^2\sqrt{1 - t^2}}, \quad x\frac{dy}{dt} = 2\frac{1 - 3t^2}{(1 + t^2)^3}.$$

So

$$A = \int_{\vec{\sigma}} x\, dy = \int_{-1}^{1} 2\frac{1 - 3t^2}{(1 + t^2)^3}\, dt = 1. \ \square$$

5. For what closed curve C is

$$\int_C y^3\, dx + (3x - x^3)\, dy$$

largest? What is the largest value?

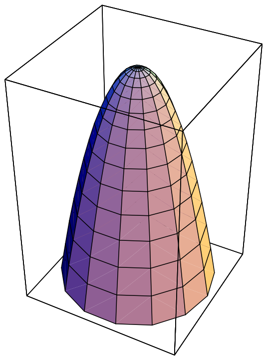
By Green's theorem,

$$\int_C y^3\, dx + (3x - x^3)\, dy = \int\int_D (3 - 3x^2 - 3y^2)\, dx\, dy,$$

where D is the region enclosed by C.

The double integral is the (signed) volume bounded by the upside-down paraboloid $z = 3 - 3x^2 - 3y^2$ and the x-y-plane. The paraboloid intersects the x-y-plane in the unit circle $x^2 + y^2 = 1$; it lies above the plane for $x^2 + y^2 < 1$ and below the plane for $x^2 + y^2 > 1$.

To get the largest volume (i.e. the maximum value for the double integral), take the volume under the positive part of the surface. To do this, integrate over the unit disk $x^2 + y^2 \le 1$, for which $C = \{x^2 + y^2 = 1\}$.



Convert to polar, so the limits are

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

The integral is

$$\int_0^1 \int_0^{2\pi} (3 - 3r^2) r\, d\theta\, dr = 2\pi \int_0^1 (3r - 3r^3)\, dr = 2\pi \left[\frac{3}{2}r^2 - \frac{3}{4}r^4\right]_0^1 = \frac{3\pi}{2} \approx 4.71239.$$

The line integral is largest if C is the unit circle (traversed counterclockwise). The maximum value is $\dfrac{3\pi}{2} \approx 4.71239$. $\square$

6. Compute

$$\int_C (xy^2z^2 + x^2y^3)\, dx + (x^2yz^2 + x^3y^2)\, dy + x^2y^2z\, dz,$$

where C is a path from $(1, 1, 1)$ to $(0, 0, 0)$.

Since the path was not specified, it is natural to assume *that it doesn't matter* — that is, that the line integral is path independent. If that is so, it should be possible to find a potential function.

A potential function $f(x, y, z)$ satisfies

$$\nabla f = \left\langle \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right\rangle = \langle xy^2z^2 + x^2y^3, x^2yz^2 + x^3y^2, x^2y^2z \rangle.$$

Equate the x-components and integrate:

$$\frac{\partial f}{\partial x} = xy^2z^2 + x^2y^3, \quad f = \int (xy^2z^2 + x^2y^3)\, dx = \frac{1}{2}x^2y^2z^2 + \frac{1}{3}x^3y^3 + g(y, z).$$

(Since the integral is with respect to x, y and z are constant, and must be included in the arbitrary constant $g(y, z)$.)

Compute $\dfrac{\partial f}{\partial y}$ and set it equal to $x^2yz^2 + x^3y^2$ (from the first f equation):

$$x^2yz^2 + x^3y^2 = \frac{\partial f}{\partial y} = x^2yz^2 + x^3y^2 + \frac{\partial g}{\partial y}, \quad \frac{\partial g}{\partial y} = 0.$$

This means that g is constant with respect to y, so it is a function of z alone: $g = h(z)$.

Now

$$f = \frac{1}{2}x^2y^2z^2 + \frac{1}{3}x^3y^3 + h(z).$$

Compute $\dfrac{\partial f}{\partial z}$ and set it equal to x^2y^2z (from the first f equation):

$$x^2y^2z = \frac{\partial f}{\partial z} = x^2y^2z + \frac{dh}{dz}, \quad \frac{dh}{dz} = 0.$$

This time, h is a numerical constant: $h = C$. At this point, h is a numerical constant C. Since the derivative of a constant is 0, and since I just need *some* potential function f, I may take $C = 0$. Then $h = 0$, so $f = \dfrac{1}{2}x^2y^2z^2 + \dfrac{1}{3}x^3y^3$.

By path independence, the value of the line integral is given by computing the potential function at the endpoints of the path and subtracting:

$$\int_C (xy^2z^2 + x^2y^3)\, dx + (x^2yz^2 + x^3y^2)\, dy + x^2y^2z\, dz = f(0, 0, 0) - f(1, 1, 1) = 0 - \left(\frac{1}{2} + \frac{1}{3}\right) = -\frac{5}{6}. \square$$

7. A particle moves through the force field $\vec{F} = \langle x^2 - y^2, 2xy \rangle$.

(a) Compute the work done by the particle in moving from $(0, 0)$ to $(2, 4)$ along the path $\vec{\sigma}(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$.

First, compute the work done by the particle in moving from $(0, 0)$ to $(2, 4)$ along the path $\vec{\sigma}(t) = \langle t, t^2 \rangle$, $0 \le t \le 2$.

$$\vec{\sigma}'(t) = \langle 1, 2t \rangle \text{ and } \vec{F}(\vec{\sigma}(t)) = \langle t^2 - t^4, 2t^3 \rangle.$$

So

$$\int_{\vec{\sigma}} \vec{F} \cdot \vec{ds} = \int_0^2 (t^2 - t^4 + 4t^4)\, dt = \int_0^2 (t^2 + 3t^4)\, dt = \left[\frac{1}{3}t^3 + \frac{3}{5}t^5\right]_0^2 = \frac{328}{15} \approx 21.86667. \square$$

(b) Compute the work done by the particle in moving from $(0, 0)$ to $(2, 4)$ along the path $\vec{\phi}(t) = \langle t, 2t \rangle$, $0 \le t \le 2$.

$$\vec{\phi}'(t) = \langle 1, 2 \rangle \text{ and } \vec{F}\left(\vec{\phi}(t)\right) = \langle -3t^2, 4t^2 \rangle.$$

So

$$\int_{\vec{\phi}} \vec{F} \cdot \vec{ds} = \int_0^2 5t^2\, dt = \left[\frac{5}{3}t^3\right]_0^2 = \frac{40}{3} \approx 13.33333. \square$$

(c) Why do you get different results in moving from $(0, 0)$ to $(2, 4)$ along different paths?

The force field $\vec{F} = \langle x^2 - y^2, 2xy \rangle$ is not a gradient field. There is no reason why different paths should give the same result if the field is not conservative. $\square$

8. Compute the work done by the particle moving through the field

$$\vec{F} = \langle 4x^3y - xy^2 + \frac{1}{x^2 + 1}, 6x + x^4 - x^2y + y^{10} \rangle$$

counterclockwise around the unit circle $\vec{\sigma}(t) = \langle \cos t, \sin t \rangle$, $0 \le t \le 2\pi$.

The work is given by the integral

$$\int_{\vec{\sigma}} \left(4x^3y - xy^2 + \frac{1}{x^2 + 1}\right) dx + (6x + x^4 - x^2y + y^{10})\, dy.$$

Since the integral looks rather messy, and since the curve is a *closed* curve, it's natural to try Green's theorem. $P = 4x^3y - xy^2 + \dfrac{1}{x^2 + 1}$ and $Q = 6x + x^4 - x^2y + y^{10}$, so

$$\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} = 6 + 4x^3 - 2xy - 4x^3 + 2xy = 6.$$

The integral is

$$\int_{\vec{\sigma}} \left(4x^3y - xy^2 + \frac{1}{x^2 + 1}\right) dx + (6x + x^4 - x^2y + y^{10})\, dy =$$

$$\iint_{x^2+y^2 \le 1} 6\, dx\, dy = 6 \cdot (\text{area of } x^2 + y^2 \le 1) = 6\pi \approx 18.84956. \square$$

©1993 by Bruce Ikenaga