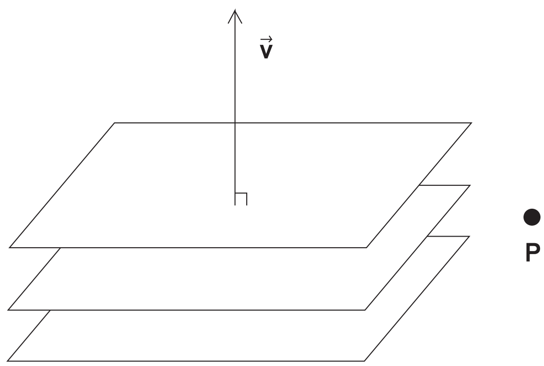
Planes.

A plane in space is determined by **a point on the plane** and **a vector perpendicular to the plane**.



As the picture shows, there are many planes in space perpendicular to a given vector $\vec{v}$. Think of the vector as a "fork" sticking in a stack of pancakes. However, there is exactly one plane which is perpendicular to $\vec{v}$ and which passes through the point P: namely, the "pancake" at the same "level" as P.

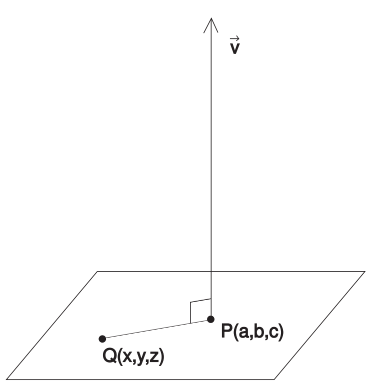
> A line in space is determined by a point on the line and a vector *parallel* to the line. People often get the line and plane conditions confused. Draw some pictures and convince yourself that there *isn't* a *unique* plane which passes through a given point and is *parallel* to a given vector.

Suppose we're thinking of the plane which passes through the point $P(a, b, c)$ and is perpendicular to the vector $\vec{v} = \langle v_1, v_2, v_3 \rangle$. What is the equation of the plane?

As the next picture shows, a point $Q(x, y, z)$ is on the plane if and only if the vector from P to Q is perpendicular to $\vec{v}$. The vector vector from $P(a, b, c)$ to $Q(x, y, z)$ is $\langle x - a, y - b, z - c \rangle$. Therefore, we must have

$$\langle v_1, v_2, v_3 \rangle \cdot \langle x - a, y - b, z - c \rangle = 0, \quad \text{or} \quad v_1(x - a) + v_2(y - b) + v_3(z - c) = 0.$$

This is the equation of the plane which passes through $P(a, b, c)$ and is perpendicular to $\langle v_1, v_2, v_3 \rangle$.



Remarks.

1. A plane is a 2-dimensional object in space. To go from 3 dimensions to 2, you need 1 condition, i.e. 1 equation. A line, on the other hand, is a 1-dimensional object in space.

To go from 3 dimensions to 1, you need 2 conditions, i.e. 2 equations. The parametric equations for a line are

$$x - a = v_1 t, \quad y - b = v_2 t, \quad z - c = v_3 t,$$

and if you eliminate t, you do indeed get 2 equations. This reasoning explains why you do not have a single (scalar) equation for a line in space.

2. When you're writing down the equation of a plane, remember that it *is* an *equation*. Don't forget the "$= 0$" on the right side!

Examples.

1. Find the equation of the plane that passes through $(1, 2, 6)$ and is perpendicular to the vector $\langle -3, 6, 11\rangle$.

We simply insert the point and the vector into the standard form:

$$-3(x-1) + 6(y-2) + 11(z-6) = 0, \quad \text{or} \quad -3x + 6y + 11z = 75.$$

In general, an equation of the form

$$Ax + By + Cz + D = 0$$

determines a plane. □

2. Find a point on the plane $2x - y - 5z = 10$. Find a vector perpendicular to the plane.

To find a point on the plane, set two of the variables equal to (randomly chosen) numbers, and solve for the third. For example, if we set $x = 0$ and $y = 0$, we obtain $-5z = 10$, or $z = -2$. Therefore, the point $(0, 0, -2)$ is on the plane.

We can obtain a vector perpendicular to the plane by "reading off" the coefficients of x, y, and z. The vector $\langle 2, -1, -5\rangle$ is perpendicular to the plane. □

3. Find the line of intersection of the planes

$$x - y + 2z = 5 \quad \text{and} \quad 2x + 3y - z = 0.$$

Method 1: We find where things intersect by solving their equations simultaneously. Row reduce the coefficient matrix for the two equations:

$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 2 & 3 & -1 & 0 \end{bmatrix} \underset{r_2 \to r_2 - 2r_1}{\longrightarrow} \begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 5 & -5 & -10 \end{bmatrix} \underset{r_2 \to \frac{1}{5} r_2}{\longrightarrow}$$
$$\begin{bmatrix} 1 & -1 & 2 & 5 \\ 0 & 1 & -1 & -2 \end{bmatrix} \underset{r_1 \to r_1 + r_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & 1 & 3 \\ 0 & 1 & -1 & -2 \end{bmatrix}$$

We get $x = -z + 3$ and $y = z + 2$. Set $z = t$. The parametric equations for the line of intersection are

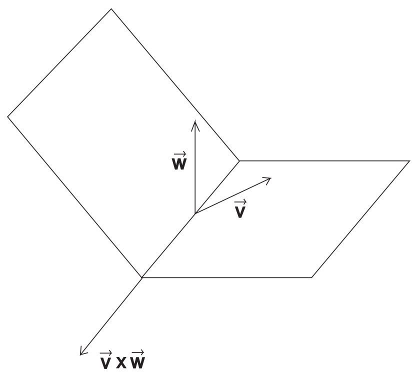
$$x = -t + 3, \quad y = t + 2, \quad z = t. \ \square$$

Method 2: A line is determined by a point on the line and a vector parallel to the line. We can determine the line of intersection by finding a point and a vector.

To find a point on the line, we must find a point which lies on both planes. We can't hope to solve for a unique point, because there are infinitely many points in the line of intersection. Instead, set $z = 0$ (to pick a value at random). The equations become $x - y = 5$ and $2x + 3y = 0$.

Solve simultaneously. We get $x = 3$, $y = -2$. Therefore, the point $(3, -2, 0)$ is on the line of intersection.

Next, we need a vector parallel to the line of intersection. Think of the two planes as the covers of a book that has been opened part of the way. Then the line of intersection is the book's spine.



As the picture shows, if we take the cross product of the vectors perpendicular to the two planes, we'll get a vector parallel to the line of intersection.

The vector $\langle 1, -1, 2\rangle$ is perpendicular to the first plane. The vector $\langle 2, 3, -1\rangle$ is perpendicular to the second plane.

$$\langle 1, -1, 2\rangle \times \langle 2, 3, -1\rangle = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & -1 & 2 \\ 2 & 3 & -1 \end{vmatrix} = \langle -5, 5, 5\rangle.$$

The line passes through the point $(3, -2, 0)$ and is parallel to the vector $\langle -5, 5, 5\rangle$. Therefore, the line is

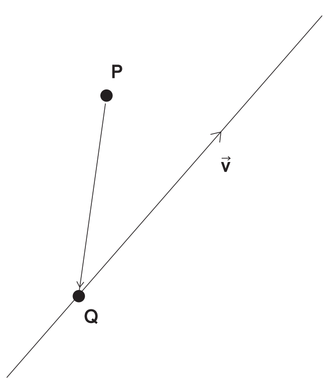
$$x - 3 = -5t, \quad y + 2 = 5t, \quad z = 5t. \square$$

4. Find the equation of the plane containing the point $P(0, 1, 1)$ and the line

$$x = 1 + t, \quad y = 1 - t, \quad z = 4t.$$

We need a point on the plane; $P(0, 1, 1)$ will do. We need a vector perpendicular to the plane. We have a construction which produces vectors perpendicular to other vectors, namely, the cross product. We'll get a vector perpendicular to the plane if we take the cross product of two vectors in the plane.

Since the line lies in the plane, its parallel vector $\langle 1, -1, 4\rangle$ lies in the plane. We need a second vector.



As the picture shows, we can get a second vector by finding a point Q on the line and using the vector $\vec{PQ}$. Set $t = 0$: We obtain the point $Q(1, 1, 0)$. Therefore, $\vec{PQ} = \langle 1, 0, -1\rangle$.

We compute the cross product:

$$\langle 1,-1,4\rangle \times \langle 1,0,-1\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -1 & 4 \\ 1 & 0 & -1 \end{vmatrix} = \langle 1,5,1\rangle.$$

Thus, the vector $\langle 1,5,1\rangle$ is perpendicular to the plane. Since the point $P(0,1,1)$ is on the plane, the plane is

$$(1)(x-0)+(5)(y-1)+(1)(z-1)=0, \quad \text{or} \quad x+5y+z=6. \ \square$$

Problems on Planes.

1. Find the equation of the plane:

(a) Which is perpendicular to the vector $\langle 2,1,3\rangle$ and passes through the point $(1,1,7)$.

(b) Which passes through the point $(4,-5,-1)$ and is *perpendicular* to the line

$$x=3-t, \quad y=4+4t, \quad z=9.$$

2. Find the point of intersection of the line

$$x=3+t, \quad y=5+2t, \quad z=2-2t$$

and the plane $2x+y-z=3$.

3. (a) Find the point on the plane $2x+y+3z=6$ closest to the origin.

> You can do this using vectors by constructing a line which is perpendicular to the plane and passes through the origin. The desired point is the point where the line intersects the plane. Alternatively, the square of the distance from (x,y,z) to the origin is
>
> $$S=x^2+y^2+z^2.$$
>
> Solve for z in the plane equation and substitute. This gives S in terms of x and y, and you can use our two-variable max-min stuff to find the closest point.

(b) Find the distance from the origin to the plane $2x+y+3z=6$.

4. Find the equation of the plane which goes through the point $(1,3,-1)$ and is parallel to the plane $3x-2y+6z=9$.

5. Find the equation of the line of intersection of the planes

$$2x-3y+z=0 \quad \text{and} \quad x+y+z=4.$$

6. Find the distance from the point $(-6,2,3)$ to the plane $4x-5y+8z=7$.

7. Find the equation of the plane containing the points $P(4,-3,1)$, $Q(6,-4,7)$, and $R(1,2,2)$.

8. (a) Show that the following lines intersect:

$$x=2+3t, \quad y=-4-2t, \quad z=-1+4t$$
$$x=6+4s, \quad y=-2+2s, \quad z=-3-2s$$

(b) Find the equation of the plane containing the two lines in part (a).

9. The lines

$$x=3+t, \quad y=2-4t, \quad z=t$$
$$x=4-s, \quad y=3+s, \quad z=-2+3s$$

are **skew**: They aren't parallel, and they don't intersect. Find the distance between the lines.

Solutions to the Problems on Planes.

1. Find the equation of the plane:

(a) Which is perpendicular to the vector $\langle 2,1,3\rangle$ and passes through the point $(1,1,7)$.

$$2(x-1)+(y-1)+3(z-7)=0, \quad \text{or} \quad 2x+y+3z=24. \square$$

(b) Which passes through the point $(4,-5,-1)$ and is *perpendicular* to the line

$$x=3-t, \quad y=4+4t, \quad z=9.$$

We need a point on the plane. Use $(4,-5,-1)$.

We need a vector perpendicular to the plane. The vector $\langle -1,4,0\rangle$ is parallel to the line, hence it is perpendicular to the plane. Therefore, the plane is

$$-(x-4)+4(y+5)=0, \quad \text{or} \quad -x+4y+24=0. \square$$

2. Find the point of intersection of the line

$$x=3+t, \quad y=5+2t, \quad z=2-2t$$

and the plane $2x+y-z=3$.

Plug the expressions for x, y, and z from the line into the equation of the plane and solve for t:

$$2(3+t)+(5+2t)-(2-2t)=3, \quad t=-1.$$

Substitute $t=-1$ into the equations for x, y, and z to obtain $x=2$, $y=3$, $z=4$. The point of intersection is $(2,3,4)$. $\square$

3. (a) Find the point on the plane $2x+y+3z=6$ closest to the origin.

The vector $\langle 2,1,3\rangle$ is perpendicular to the plane. The line which passes through the origin and is perpendicular to the plane is

$$x=2t, \quad y=t, \quad z=3t.$$

We find the point where the line intersects the plane by substituting these expressions into the plane equation:

$$2\cdot 2t+t+3\cdot 3t=6, \quad t=\frac{3}{7}.$$

Plug this back into the x-y-z equations to obtain $x=\frac{6}{7}$, $y=\frac{3}{7}$, $z=\frac{9}{7}$. The point on the plane closest to the origin is $\left(\frac{6}{7},\frac{3}{7},\frac{9}{7}\right)$. $\square$

(b) Find the distance from the origin to the plane $2x+y+3z=6$.

The distance from the origin to the plane $2x+y+3z=6$ is just the distance from $\left(\frac{6}{7},\frac{3}{7},\frac{9}{7}\right)$ to the origin:

$$\sqrt{\left(\frac{6}{7}\right)^2+\left(\frac{3}{7}\right)^2+\left(\frac{9}{7}\right)^2}=\frac{\sqrt{126}}{7}\approx 1.60357. \square$$

4. Find the equation of the plane which goes through the point $(1, 3, -1)$ and is parallel to the plane $3x - 2y + 6z = 9$.

The vector $\langle 3, -2, 6\rangle$ is perpendicular to $3x - 2y + 6z = 9$. Since our plane is parallel to the given plane, our plane is also perpendicular to $\langle 3, -2, 6\rangle$. Our plane goes through the point $(1, 3, -1)$, so its equation is

$$3(x-1) - 2(y-3) + 6(z+1) = 0, \quad \text{or} \quad 3x - 2y + 6z = -6. \square$$

5. Find the equation of the line of intersection of the planes

$$2x - 3y + z = 0 \quad \text{and} \quad x + y + z = 4.$$

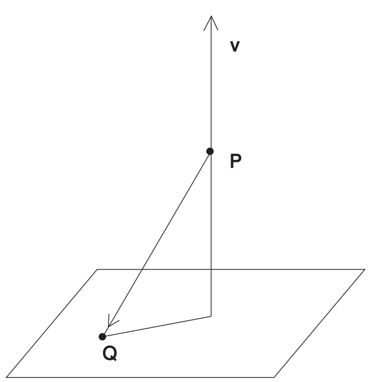
Solve the equations simultaneously by row reducing the coefficient matrix:

$$\begin{bmatrix} 2 & -3 & 1 & 0 \\ 1 & 1 & 1 & 4 \end{bmatrix} \underset{r_1 \leftrightarrow r_2}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 2 & -3 & 1 & 0 \end{bmatrix} \underset{r_2 \to r_2 - 2r_1}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & -5 & -1 & -8 \end{bmatrix}$$
$$\underset{r_2 \to -\frac{1}{5}r_2}{\longrightarrow} \begin{bmatrix} 1 & 1 & 1 & 4 \\ 0 & 1 & \frac{1}{5} & \frac{8}{5} \end{bmatrix} \underset{r_1 \to r_1 - r_2}{\longrightarrow} \begin{bmatrix} 1 & 0 & \frac{4}{5} & \frac{12}{5} \\ 0 & 1 & \frac{1}{5} & \frac{8}{5} \end{bmatrix}$$

We obtain $x = -\dfrac{4}{5}z + \dfrac{12}{5}$, $y = -\dfrac{1}{5}z + \dfrac{8}{5}$. Set $z = t$. The parametric equations for the line of intersection are

$$x = -\frac{4}{5}t + \frac{12}{5}, \quad y = -\frac{1}{5}t + \frac{8}{5}, \quad z = t. \square$$

6. Find the distance from the point $(-6, 2, 3)$ to the plane $4x - 5y + 8z = 7$.



The vector $\vec{v} = \langle 4, -5, 8\rangle$ is perpendicular to the plane. Since

$$4 \cdot 1 - 5 \cdot 1 + 8 \cdot 1 = 7,$$

the point $Q(1, 1, 1)$ is on the plane. The vector from $P(-6, 2, 3)$ to Q is $\langle 7, -1, -2\rangle$. The distance is the absolute value of the components of $\vec{PQ}$ in the direction of $\vec{v}$:

$$\operatorname{comp}_{\vec{v}} \vec{PQ} = \frac{\langle 7, -1, -2\rangle \cdot \langle 4, -5, 8\rangle}{|\langle 4, -5, 8\rangle|} = \frac{17}{\sqrt{105}} \approx 1.65703. \square$$

7. Find the equation of the plane containing the points $P(4, -3, 1)$, $Q(6, -4, 7)$, and $R(1, 2, 2)$.

The vectors $\vec{PQ} = \langle 2, -1, 6\rangle$ and $\vec{PR} = \langle -3, 5, 1\rangle$ lie in the plane, so their cross product is perpendicular to the plane:

$$\langle 2,-1,6\rangle \times \langle -3,5,1\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 2 & -1 & 6 \\ -3 & 5 & 1 \end{vmatrix} = \langle -31,-20,7\rangle.$$

Since $P(4,-3,1)$ lies on the plane, the plane is

$$-31(x-4) - 20(y+3) + 7(z-1) = 0, \quad -31x - 20y + 7z = -57. \square$$

8. (a) Show that the following lines intersect:

$$x = 2+3t, \quad y = -4-2t, \quad z = -1+4t$$
$$x = 6+4s, \quad y = -2+2s, \quad z = -3-2s$$

Equate the expressions for x: $2+3t = x = 6+4s$, so $t = \dfrac{4}{3} + \dfrac{4}{3}s$.

Equate the y-expressions and substitute for t: $-4-2t = y = -2+2s$, so $-4 - 2\left(\dfrac{4}{3} + \dfrac{4}{3}s\right) = -2+2s$, and $s = -1$. From this, we obtain $t = 0$.

Check for consistency by substituting the s and t values into the equations for z: $z = -1+4t = -1$ and $z = -3-2s = -1$.

Since z-values check, the lines intersect. $t = 0$ gives $x = 2$, $y = -4$, $z = -1$: The intersection point is $(2,-4,-1)$. $\square$

(b) Find the equation of the plane containing the two lines in part (a).

We need a point on the plane. From part (a), the point $(2,-4,-1)$ is on both lines, so it is surely in the plane.

We need a vector perpendicular to the plane. The vector $\langle 3,-2,4\rangle$ is parallel to the first line, and the vector $\langle 4,2,-2\rangle$ is parallel to the second line. Hence, the cross product is perpendicular to the plane containing the lines.

$$\langle 3,-2,4\rangle \times \langle 4,2,-2\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 3 & -2 & 4 \\ 4 & 2 & -2 \end{vmatrix} = \langle -4,22,14\rangle.$$

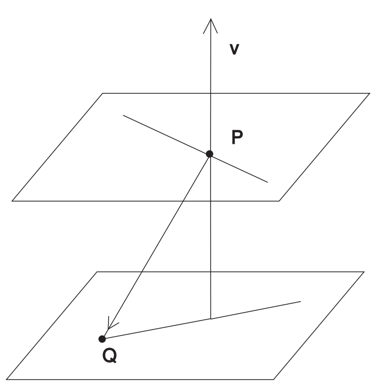
Therefore, the plane is

$$-4(x-2) + 22(y+4) + 14(z+1) = 0, \quad \text{or} \quad 2x - 11y - 7z = 55. \square$$

9. The lines

$$x = 3+t, \quad y = 2-4t, \quad z = t$$
$$x = 4-s, \quad y = 3+s, \quad z = -2+3s$$

are **skew**: They aren't parallel, and they don't intersect. Find the distance between the lines.



Picture one line lying in the ceiling and the other lying in the floor. The method we use is essentially the same as the method we'd use for the distance between parallel planes, or the distance from a point to a plane.

Set $t = 0$: The point $P(3, 2, 0)$ lies in the first line. Set $s = 0$: The point $Q(4, 3, -2)$ lies in the second line. We have $\vec{PQ} = \langle 1, 1, -2\rangle$.

Next, we need a vector perpendicular to both the "ceiling" and the "floor", i.e. perpendicular to both lines. The vector $\langle 1, -4, 1\rangle$ is parallel to the first line, and the vector $\langle -1, 1, 3\rangle$ is parallel to the second line. Hence, their cross product will be perpendicular to both lines:

$$\langle 1, -4, 1\rangle \times \langle -1, 1, 3\rangle = \begin{vmatrix} \hat{\imath} & \hat{\jmath} & \hat{k} \\ 1 & -4 & 1 \\ -1 & 1 & 3 \end{vmatrix} = \langle -13, -4, -3\rangle.$$

To find the distance, we take the absolute value of the component of $\vec{PQ}$ on $\vec{v} = \langle -13, -4, -3\rangle$:

$$\operatorname{comp}_{\vec{v}} \vec{PQ} = \frac{\langle 1, 1, -2\rangle \cdot \langle -13, -4, -3\rangle}{|\langle -13, -4, -3\rangle|} = -\frac{11}{\sqrt{194}}.$$

The distance is $\dfrac{11}{\sqrt{194}} \approx 0.78975$. □