Converting a double integral to polar coordinates.

The polar conversion equations

$$x = r\cos\theta, \quad y = r\sin\theta$$

define a function $f : \mathbf{R}^2 \longrightarrow \mathbf{R}^2$ by

$$f(r,\theta) = (r\cos\theta, r\sin\theta).$$

The Jacobian of f is

$$\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r.$$

For $r \geq 0$, the absolute value is just r. Hence, when we go from rectangular coordinates to polar coordinates, the differentials convert by

$$dx\,dy \longrightarrow r\,dr\,d\theta$$

Therefore, in order to convert a double integral from rectangular coordinates to polar coordinates, we do the following:

Step 1: Convert the limits of integration by describing the region of integration by inequalities in polar coordinates.

Step 2: Convert the integrand using the polar conversion formulas.

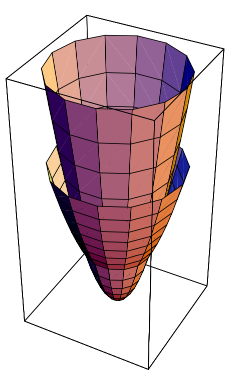
$$x = r\cos\theta, \quad y = r\sin\theta,$$
$$r^2 = x^2 + y^2, \quad \frac{y}{x} = \tan\theta.$$

Step 3: Convert the differentials using

$$dx\,dy \longrightarrow r\,dr\,d\theta$$

Example. Find the volume of the region defined by

$$2x^2 + 2y^2 \leq z \leq x^2 + y^2 + 1.$$



$z = 2x^2 + 2y^2$ and $z = x^2 + y^2 + 1$ are paraboloids. The region is bounded *below* by the first and *above* by the second, so we should be integrating

$$(x^2 + y^2 + 1) - (2x^2 + 2y^2) = 1 - x^2 - y^2$$

over the projection of the region into the x-y-plane.

To find the projection, we find the curve where the two surfaces intersect:

$$2x^2 + 2y^2 = x^2 + y^2 + 1, \quad x^2 + y^2 = 1.$$

The projection of the region is therefore the interior of the circle $x^2 + y^2 = 1$: the circle of radius 1 centered at the origin.

Since the integrand contains the expression "$x^2 + y^2$", and since we're integrating over the interior of a circle, it is reasonable to consider converting to polar.

The integrand is $1 - x^2 - y^2 = 1 - r^2$.

The inequalities

$$0 \le \theta \le 2\pi, \quad 0 \le r \le 1$$

describe the interior of the circle $x^2 + y^2 = 1$.

Therefore, the volume is

$$\int_0^{2\pi} \int_0^1 (1 - r^2) \cdot r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 d\theta = \frac{1}{4} \int_0^{2\pi} d\theta = \frac{\pi}{2} \approx 1.57080. \square$$

When should you consider converting to polar coordinates? Here are two rough guidelines:

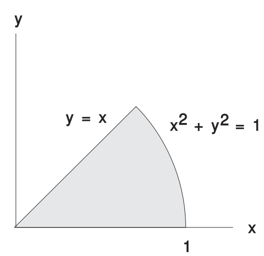
1. Consider converting to polar coordinates when the region of integration involves graphs that "look nice" in polar. **Circles** obviously "look nice" in polar. So do graphs that are usually given in polar form, such as cardioids.

2. Consider converting to polar coordinates when the integrand involves terms like $x^2 + y^2$ $(= r^2)$.

Example. Compute

$$\int_R \int 4x^2\, dx\, dy,$$

where R is the region in the first quadrant bounded above by the line $y = x$ and the semicircle $y = \sqrt{1 - x^2}$ and below by the x-axis.



The region is described by the inequalities

$$0 \le r \le 1, \quad 0 \le \theta \le \frac{\pi}{4}.$$

Therefore,

$$\int_R \int 4x^2\, dx\, dy = \int_0^1 \int_0^{\pi/4} 4r^3 (\cos\theta)^2\, d\theta\, dr = \int_0^1 2r^3 \int_0^{\pi/4} (1 + \cos 2\theta)\, d\theta\, dr =$$

$$\int_0^1 2r^3 \left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\pi/4} dr = \left(\frac{\pi}{4} + \frac{1}{2}\right) \int_0^1 2r^3\, dr = \frac{\pi}{8} + \frac{1}{4} \approx 0.64270. \square$$

Problems on converting to polar coordinates.

1. Find the volume of the region lying above the ring

$$1 \leq x^2 + y^2 \leq 4$$

in the x-y-plane and below the cone $z = \sqrt{x^2 + y^2}$.

2. Compute

$$\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (x+y)\, dy\, dx.$$

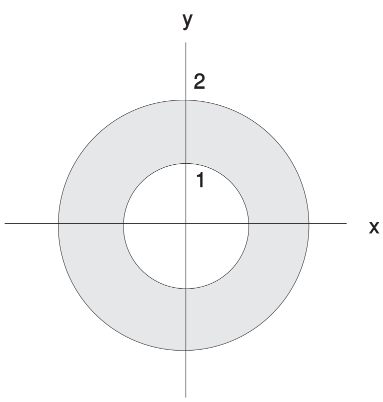
Solutions to the Problems on converting to polar coordinates.

1. Find the volume of the region lying above the ring

$$1 \leq x^2 + y^2 \leq 4$$

in the x-y-plane and below the cone $z = \sqrt{x^2 + y^2}$.

Here is a picture of the ring:



In polar coordinates, the ring is described by the inequalities

$$0 \leq \theta \leq 2\pi, \quad 1 \leq r \leq 2.$$

Since $\sqrt{x^2 + y^2} = r$, the volume is

$$\int_0^{2\pi} \int_1^2 r \cdot r\, dr\, d\theta = \int_0^{2\pi} \left[\frac{1}{3}r^3\right]_1^2 d\theta = \frac{7}{3} \int_0^{2\pi} d\theta = \frac{14\pi}{3} \approx 14.66077. \square$$

2. Compute

$$\int_0^2 \int_{-\sqrt{2x-x^2}}^{\sqrt{2x-x^2}} (x+y)\, dy\, dx.$$

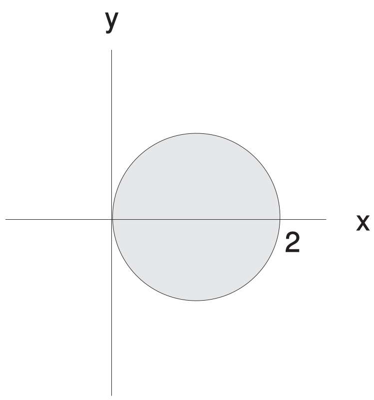
We convert the integral to polar coordinates. We will convert the limits of integration first. Pull off the limits as inequalities:

$$0 \leq x \leq 2, \quad -\sqrt{2x - x^2} \leq y \leq \sqrt{2x - x^2}.$$

Draw the region described by the inequalities. Notice that

$$y = \pm\sqrt{2x - x^2} \quad \text{so} \quad (x-1)^2 + y^2 = 1.$$

Hence, the bounding curves for y are respectively the top ($y = \sqrt{2x - x^2}$) and bottom ($y = -\sqrt{2x - x^2}$) of the circle $(x-1)^2 + y^2 = 1$, which has radius 1 and center at $(1, 0)$. The picture is therefore:



Next, we describe the region by inequalities in polar coordinates. To do this, we need the polar equation of the circle:

$$(x-1)^2 + y^2 = 1, \quad x^2 - 2x + 1 + y^2 = 1, \quad x^2 + y^2 = 2x, \quad r^2 = 2r\cos\theta, \quad r = 2\cos\theta.$$

Note that this circle is traced out once as θ goes from $-\dfrac{\pi}{2}$ to $\dfrac{\pi}{2}$. (The circle lies entirely in the right half-plane.) Hence, the inequalities are:

$$-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}, \quad 0 \le r \le 2\cos\theta.$$

The integrand is $x + y = r(\cos\theta + \sin\theta)$.
Our integral is:

$$\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \int_0^{2\cos\theta} r(\cos\theta + \sin\theta) \cdot r\,dr\,d\theta = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (\cos\theta + \sin\theta)\left[\frac{1}{3}r^3\right]_0^{2\cos\theta} d\theta =$$
$$\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((\cos\theta)^3 \sin\theta + (\cos\theta)^4\right) d\theta.$$

Here's the work for the two integrals. Do this one by substituting $u = \cos\theta$, so $du = -\sin\theta\,d\theta$:

$$\int (\cos\theta)^3 \sin\theta\,d\theta = -\frac{1}{4}(\cos\theta)^4 + C.$$

The next one requires repeated use of the double angle formula:

$$\int (\cos\theta)^4\,d\theta = \int \left(\frac{1}{2}(1 + \cos 2\theta)\right)^2 d\theta = \frac{1}{4}\int \left(1 + 2\cos 2\theta + (\cos\theta)^2\right) d\theta =$$
$$\frac{1}{4}\int \left(1 + 2\cos 2\theta + \frac{1}{2}(1 + \cos 4\theta)\right) d\theta = \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta + C.$$

Therefore, we get

$$\frac{8}{3}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left((\cos\theta)^3 \sin\theta + (\cos\theta)^4\right) d\theta = \frac{8}{3}\left[-\frac{1}{4}(\cos\theta)^4 + \frac{3}{8}\theta + \frac{1}{4}\sin 2\theta + \frac{1}{32}\sin 4\theta\right]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} = \pi \approx 3.14159. \square$$