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Spectral Approximation of Multiplication Operators

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Abstract. A linear operator on a Hilbert space may be approximated with finite matrices by choosing an orthonormal basis of the Hilbert space. For an operator that is not compact such approximations cannot converge in the norm topology on the space of operators. Multiplication operators on spaces of L_2 functions are never compact; for them we consider how well the eigenvalues of the matrices approximate the spectrum of the multiplication operator, which is the essential range of the multiplier. The choice of the orthonormal basis strongly affects the convergence. Toeplitz matrices arise when using the Fourier basis of exponentials $\exp(ik\theta)$. We also consider the basis of Legendre polynomials and the basis of Walsh functions.

Contents

1. Introduction

The idea of approximating a linear operator by finite matrices is an obvious one that must occur again and again. One can see in Fredholm's approach to integral

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equations the use of this idea, and perhaps the first use was even earlier [15, 32]. The Galerkin method applied to linear differential operators is another early use of the idea [17, 13]. This article considers the problems of approximating the spectra of multiplication operators by the spectra of finite matrices.

The introduction discusses the general problem of using finite matrices to approximate bounded operators. The second section treats multiplication operators in general and then looks at three specific bases: the Fourier basis, the basis of Legendre polynomials, and the basis of Walsh functions. The next three sections treat the three bases in more detail with summaries of the fundamental results and detailed analysis of three examples including plots of the computed eigenvalues.

Some of the results described are contained in the literature, particularly for the Fourier basis (Toeplitz operators) and the Legendre basis. An important source is the book of Gohberg and Feldman [16], which treats the question solving equations $Ax = y$, where A is an operator on a Hilbert space, by approximating A by larger and larger matrices. Encyclopedic coverage of Toeplitz operators is contained in Böttcher and Silbermann [9].

Let $A : \mathbf{H} \to \mathbf{H}$ be a bounded linear operator on a Hilbert space $\mathbf{H}$. While unbounded operators are certainly interesting and of special importance in mathematical physics, the approximation questions we consider are still significant even for bounded operators. We use the operator norm

$$\|A\| = \sup_{\|x\|=1} \|Ax\|.$$

Let $e_0, e_1, e_2, \ldots$ be an orthonormal basis of $\mathbf{H}$. The operator A is represented by the infinite matrix, also denoted by A, whose ij entry is $a_{ij} = \langle Ae_j, e_i \rangle$. By taking the upper left $n \times n$ corner of A we get finite matrix, denoted by A_n, that we can regard as a matrix approximation of A, and one that we expect to somehow be a better approximation as n grows larger.

Let $P_n : \mathbf{H} \to \mathbf{H}$ be the projection onto the span of the first n basis elements $e_0, \ldots, e_{n-1}$. The matrix A_n represents the operator $P_n A$ restricted to the image of P_n, and it is convenient to use the same notation for the matrix and the operator. The operator $P_n A P_n$ is represented by the block form

$$\begin{bmatrix} A_n & 0 \\ 0 & 0 \end{bmatrix}.$$

Although we want to compare A_n and A, it is the operators $P_n A P_n$ and A that we can compare.

Recall that a sequence of linear operators $B_n : \mathbf{H} \to \mathbf{H}$ *converges strongly* to B if for each $x \in \mathbf{H}$ the sequence $B_n x$ converges to Bx. Now, for a bounded operator A and orthonormal basis $e_0, e_1, e_2, \ldots$ we have a sequence of finite rank operators A_n and two notions of convergence: norm and strong. For strong convergence we have this elementary, yet fundamental, result.

Theorem 1.1. *If A is bounded then A_n converges strongly to A.*

Proof. Left to the reader. □

Recall that the compact operators (on a Hilbert space) are the norm closure of the finite rank operators. See, for example, Bonic [8, p. 77] or Douglas [11, p. 124]. For norm convergence we have the next result.

Theorem 1.2. *A_n converges to A in norm if and only if A is compact.*

Proof. One direction follows from the characterization of the compact operators as the closure of the finite rank operators, since A_n is a sequence of finite rank operators. The other direction can be proved by first showing that it is true for finite rank operators and then using a sequence of finite rank operators that approaches the compact operator. □

This settles the question of approximation by finite matrices in the norm topology: only for compact operators is the procedure valid. On the other hand, in the strong operator topology the approximation always works. So it is between these extremes that we pursue the natural question of approximating the spectrum of A by the eigenvalues of A_n. It is the case that for a generic matrix (namely the diagonalizable or semi-simple matrices), the eigenvalues characterize the matrix up to equivalence. And so for an operator A we may find it quite satisfying that the eigenvalues of A_n have a limit in some sense that is the spectrum of A, although the sequence of operators does not converge in norm. We can borrow some motivation from quantum mechanics in which the knowledge of the spectrum of the Schrödinger operator is the primary goal and for which the approximation of the operator by matrices may give reliable information.

For the remainder of the paper we will consider multiplication operators acting on the L_2 functions on an interval. Although the operators themselves are easy to understand, the spectral approximation questions are not. One especially important case is that of Toeplitz matrices. They arise naturally when we use the orthogonal basis $\varepsilon_n(\theta) = \exp(in\theta)$, for $n \in \mathbf{Z}$, for the Hilbert space $L_2[0, 2\pi]$. The asymptotic distribution of the eigenvalues of growing Toeplitz matrices has applications in statistical mechanics and signal processing, and much of the impetus for studying the eigenvalue asymptotics comes from questions outside mathematics. See [14, 6, 21]. But this orthogonal basis is only one of many that we might use. Two others that we will consider are the Walsh functions and the Legendre polynomials. These cases have not been explored in nearly the detail as the Toeplitz matrices, and so they provide an area in which much remains to be discovered. Some of the recent discoveries about Toeplitz matrix eigenvalues have been discovered with the aid of computer programs and some of the results we will show about the other orthogonal bases will be numerical and graphical.

When the function ϕ is real-valued, the multiplication operator $M[\phi]$ is Hermitian or self-adjoint. One defines an Hermitian form on the Hilbert space $L_2[a, b]$ by $\langle M[\phi]f, g\rangle$. The finite matrices $M_n[\phi]$ are also Hermitian and hence have real eigenvalues and they define Hermitian forms on finite dimensional spaces. In this context the questions raised in this paper have been treated in work by Szegö, notably in [18]. Much of the work in this field since then has extended the results, with necessary modifications, from the case of real generating functions to the case of complex generating functions, or in the language of operator theory, from the self-adjoint to the non-self-adjoint setting. The problems turn out to be harder, just as eigenvalue questions for general matrices are harder than for self-adjoint matrices, although the intuition remains the same, since it is not at all obvious why multiplication by a (non-real) complex valued function is any more difficult to understand than multiplication by a real function.

2. Multiplication operators

From now on, we stay in the following setting. Suppose $I \subset \mathbf{R}$ is an interval and $\phi : I \to \mathbf{C}$ is a bounded measurable function. Define the multiplication operator $M[\phi] : L_2(I) \to L_2(I) : f \mapsto \phi f$. Let $e_0, e_1, e_2, \dots, e_n, \dots$ be an orthonormal basis of $L_2(I)$. (The indexing of the basis will not always be the same.) We define the $n \times n$ matrix $M_n[\phi] = (a_{ij})$, $0 \le i, j \le n-1$, where $a_{ij} = \langle M[\phi]e_j, e_i \rangle$. Thus, $a_{ij} = \int_I \phi(x) e_j(x) \overline{e_i(x)} dx$. The infinite matrix (a_{ij}), $0 \le i, j$, represents the operator $M[\phi]$.

The map M is an algebra homomorphism from the algebra of L_∞ functions on I to the algebra of bounded linear operators on $L_2(I)$, and, moreover, M is norm-preserving where the norm of a function is its essential supremum.

Theorem 2.1. *Let ϕ be a bounded measurable function. The multiplication operator $M[\phi]$ is compact if and only if $\phi = 0$ almost everywhere.*

Proof. Left to the reader. $\square$

2.1. Toeplitz Matrices.

In this example $I = [0, 2\pi]$ and the orthonormal basis is $\exp(ik\theta)$, $k \in \mathbf{Z}$. The jk entry of the matrix representation is

$$\begin{aligned} a_{jk} &= \langle M[\phi]e_k, e_j \rangle \\ &= \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) \exp(ik\theta) \overline{\exp(ij\theta)}\, d\theta \\ &= = \frac{1}{2\pi} \int_0^{2\pi} \phi(\theta) \exp(i(k-j)\theta)\, d\theta \end{aligned}$$

which is the $(j-k)$th Fourier coefficient of ϕ. Let the Fourier series of ϕ be given by $\phi(\theta) = \sum_{k \in \mathbf{Z}} c_k e^{ik\theta}$. The operator $M[\phi]$ is represented by the doubly infinite Toeplitz matrix (c_{j-k}), where j and k range over all integers. The finite Toeplitz matrix of size $n \times n$

$$\begin{bmatrix} c_0 & c_{-1} & c_{-2} & \cdots & c_{-n+1} \\ c_1 & c_0 & c_{-1} & \cdots & c_{-n+2} \\ c_2 & c_1 & c_0 & \cdots & c_{-n+3} \\ \vdots & \vdots & \vdots & & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix} \tag{1}$$

is the matrix of the restriction of $M[\phi]$ to the span of e_k, where $-n/2+1 \le k \le n/2$, for n even, or $-n/2+1 \le k \le n/2-1$, for n odd. In this way as n grows we will get an improving approximation to $M[\phi]$. It is conventional to denote the Toeplitz matrices associated to ϕ by $T_n[\phi]$.

2.2. Matrices Associated to Legendre Polynomials.

On the interval $I = [-1, 1]$ the Legendre polynomials are orthogonal with the usual inner product. Using the usual definition they are not normalized. The first five Legendre polynomials are

$$
\begin{aligned}
P_0(x) &= 1\\
P_1(x) &= x\\
P_2(x) &= \frac{3x^2-1}{2}\\
P_3(x) &= \frac{5x^3-3x}{2}\\
P_4(x) &= \frac{35x^4-30x^2+3}{8}
\end{aligned}
$$

Since $\langle P_n, P_n\rangle = 2/(2n+1)$ we multiply P_n by $\sqrt{(2n+1)/2}$ in order to normalize it. Let p_n denote the normalized Legendre polynomial $\sqrt{(2n+1)/2}P_n$. (As an aside, it is useful to keep in mind that normalization is merely a convenience. Changing the basis elements by scaling them does not change the spectrum of the finite matrices, because they are related by a change of basis, which is, in fact, a diagonal matrix.) For a function $\phi : [-1,1] \to \mathbf{C}$ the multiplication operator $M[\phi]$ has a symmetric matrix representation (with complex entries) whose ij coefficient is

$$
\frac{\sqrt{(2i+1)(2j+1)}}{2}\int_{-1}^{1}\phi(x)P_i(x)P_j(x)dx \tag{2}
$$

2.3. Walsh-Toeplitz Matrices. Let I be the interval $[0,1]$. The Walsh functions are an orthonormal basis $\psi_0, \psi_1, \dots, \psi_n, \dots$ of step functions taking on only the values 1 and -1 [29, 12]. The graphs of the first eight are shown in Figure 1.

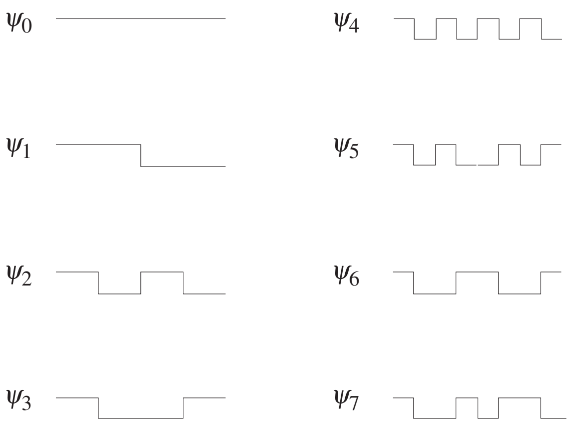

FIGURE 1. Walsh functions.

They are often described as discrete analogues of the sine and cosine functions, and, as such, have found applications in electrical engineering for digital signal processing [20, 7]. Recent interest in wavelets has focused attention on the closely related Haar functions [25, 22]. There are different ways to describe the Walsh functions and there are alternative indexing conventions. For the purposes of this paper the most convenient construction of the Walsh functions is obtained from the increasing sequence of subspaces $\mathbf{H}_m \subset \mathbf{H}_{m+1}$ of $L_2[0,1]$, $m \in \mathbf{N}$, where $\mathbf{H}_m$

is the subspace of functions that are constant on the intervals of the partition of I into intervals of length $1/2^m$. The dimension of $\mathbf{H}_m$ is 2^m. We define a basis $\psi_0, \psi_1, \dots, \psi_{2^m-1}$ of $\mathbf{H}_m$ by describing the value of ψ_i on each subinterval. We identify a subinterval by its left endpoint x, which has the form $x = \sum_{j=1}^m b_k 2^{-k}$, where each b_k is 0 or 1. Represent this endpoint by the sequence $b = (b_1, b_2, \dots, b_m)$. Express an integer i in binary form $i = \alpha_0 + \alpha_1 2 + \ldots + \alpha_{m-1} 2^{m-1}$ and represent it by the sequence $\alpha = (\alpha_0, \dots, \alpha_{m-1})$. Consider b as an element of the group $\mathbf{Z}_2{}^m$ and α as a character of the group. The value of the character α on b is

$$\alpha(b) = (-1)^{\alpha \cdot b} = (-1)^{\sum \alpha_k b_{k+1}} \tag{3}$$

This we define to be the value of $\psi_i(x)$:

$$\psi_i(x) = (-1)^{\alpha \cdot b}. \tag{4}$$

Since characters are orthogonal and since the inner product in $\mathbf{H}_m$ is the same as the inner product in the space of functions on $\mathbf{Z}_2{}^m$, we have produced an orthogonal basis of $\mathbf{H}_m$. It is, in fact, an orthonormal basis. The span of $\{\psi_n \mid n \in \mathbf{N}\}$ is the union of the subspaces $\mathbf{H}_m$ and is easily seen to be dense in $L_2[0,1]$.

Let α and α' be the sequences associated to the integers i and j. Then the product $\psi_i \psi_j$ is another basis element ψ_k, where k has the binary sequence $\alpha + \alpha'$, the sum being taken in $\mathbf{Z}_2{}^m$. We call k the *dyadic sum* of i and j and write it as $i \oplus j$. Thus, we have

$$\psi_i \psi_j = \psi_{i \oplus j} \tag{5}$$

The natural numbers with dyadic sum and the Walsh functions with multiplication are groups isomorphic to the countable sum of copies of $\mathbf{Z}_2$. Notice that the property of the basis forming a group is shared by the Fourier basis $e^{in\theta}$, but not by the basis of Legendre polynomials.

From the point of view of harmonic analysis on groups, the Fourier series representation sets up an isomorphism between the Hilbert spaces $L_2(\mathbf{T})$ and $l_2(\mathbf{Z})$, where the group of integers $\mathbf{Z}$ is the character group of the circle group $\mathbf{T}$ and vice-versa. The Walsh basis is a group isomorphic to the direct sum (coproduct) of countably many copies of $\mathbf{Z}_2$. The elements are binary sequences $(b_1, b_2, \dots, b_n, \dots)$ with only a finite number of 1's. This group is the character group of the countable product of copies of $\mathbf{Z}_2$. At first this group does not look like the interval $[0,1]$, and, topologically, they are quite different. However the map

$$\mathbf{Z}_2{}^{\mathbf{N}} \to [0,1] : b \mapsto \sum b_k 2^{-k} \tag{6}$$

is surjective and, except for a set of measure zero, it is also injective. The difficulty lies with the numbers that do not have a unique binary representation. The map is 2 to 1 onto this subset. If we give $\mathbf{Z}_2{}^{\mathbf{N}}$ the product measure arising from the measure on $\mathbf{Z}_2$ that has mass $1/2$ on each element, then (6) sets up an isomorphism of measure spaces, and therefore an isomorphism (in fact, isometry) of L_2 spaces.

The Walsh basis, like the Fourier basis, has an associated fast transform. That is, the computation of the coefficients $a_0, \dots, a_{2^m-1}$ for a function $f = \sum_{k=0}^{2^m-1} a_k \psi_k$ can be done with a number of steps on the order of $n \log_2 n$, where $n = 2^m$. In fact, the multiplications are all by 1 or -1, making it even more efficient than the fast Fourier transform, which uses nth roots of unity.

Now consider a function $\phi : [0,1] \to \mathbf{C}$ and the matrices $M_n[\phi]$. The ij entry is

$$\int_0^1 \phi(x)\psi_j(x)\psi_i(x)dx = \int_0^1 \phi(x)\psi_{i\oplus j}(x)dx = a_{i\oplus j}$$

where $\phi = \sum_{i=0}^{\infty} a_i\psi_i$ is the Walsh expansion of ϕ. Thus, similar to the Toeplitz matrices, these **Walsh-Toeplitz matrices**, as we shall call them, have coefficients that depend on the dyadic difference, which is the same as the sum, of the i and j. Notice that they are symmetric matrices, as well. It is not so easy to recognize a Walsh-Toeplitz matrix as it is to recognize a Toeplitz matrix. Below is a general 8 by 8 Walsh-Toeplitz matrix. The upper left corners show the Walsh-Toeplitz matrices of smaller order. (The rows and columns are indexed from 0 to 7.)

$$\begin{bmatrix} a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7 \\ a_1 & a_0 & a_3 & a_2 & a_5 & a_4 & a_7 & a_6 \\ a_2 & a_3 & a_0 & a_1 & a_6 & a_7 & a_4 & a_5 \\ a_3 & a_2 & a_1 & a_0 & a_7 & a_6 & a_5 & a_4 \\ a_4 & a_5 & a_6 & a_7 & a_0 & a_1 & a_2 & a_3 \\ a_5 & a_4 & a_7 & a_6 & a_1 & a_0 & a_3 & a_2 \\ a_6 & a_7 & a_4 & a_5 & a_2 & a_3 & a_0 & a_1 \\ a_7 & a_6 & a_5 & a_4 & a_3 & a_2 & a_1 & a_0 \end{bmatrix} \tag{7}$$

This 8 by 8 matrix depends on 8 parameters, whereas an 8 by 8 Toeplitz matrix depends on 15 parameters. (An 8 by 8 symmetric Toeplitz matrix depends on 8 parameters.) Notice that the 7 by 7 Walsh-Toeplitz matrix also depends on 8 parameters, as do the 5 by 5 and 6 by 6. In general, for an n by n Walsh matrix, where $2^{m-1} < n \leq 2^m$, there are 2^m coefficients that appear in the matrix.

3. Spectral Convergence

Recall that the spectrum of a linear operator A consists of those complex numbers λ such that $A - \lambda I$ is not invertible. For a linear map on a finite dimensional space everything in the spectrum is an eigenvalue, but in infinite dimensions the operator $A - \lambda I$ can fail to be invertible for a number of reasons: it may be injective but not surjective, surjective but not injective, or the inverse may be unbounded. It is straightforward to identify the spectrum of a multiplication operator with the essential range of the function.

Definition 3.1. Let (X, μ) be a measure space and $\phi : X \to \mathbf{C}$ a measurable function. The *essential range* of ϕ, denoted by $\mathcal{R}[\phi]$, is the set of complex numbers w such that $\mu(\{z \in X : |\phi(z) - w| < \epsilon\}) > 0$ for every $\epsilon > 0$.

Theorem 3.2. *Let (X, μ) be a measure space and $\phi : X \to \mathbf{C}$ an L_∞ function. Then the spectrum of the multiplication operator $M[\phi]$ on $L_2(X, \mu)$ is the essential range of ϕ.*

Proof. Left to the reader. □

Perhaps the most straightforward notion of convergence is that of the sets of eigenvalues of the matrices $M_n[\phi]$ to the essential range of ϕ.

Definition 3.3. The set of compact subsets of $\mathbf{C}$ is $\mathcal{H}(\mathbf{C})$, the *Hausdorff space* of $\mathbf{C}$. Define the Hausdorff metric $h(X,Y) = \max\{d(X,Y), d(Y,X)\}$, where

$$d(X,Y) = \sup_{x\in X} \inf_{y\in Y} |x-y| \tag{8}$$

The distance between the sets X and Y is less than ϵ exactly when X and Y are each within the ϵ-neighborhoods of each other.

Let $\Lambda_n[\phi]$ be the set of eigenvalues of $M_n[\phi]$ and consider $\Lambda_n[\phi]$ as an element of $\mathcal{H}(\mathbf{C})$. Let $\mathcal{R}[\phi]$ denote the essential range of ϕ, also an element of $\mathcal{H}(\mathbf{C})$. One might ask if the sequence $\Lambda_n[\phi]$ converges to $\mathcal{R}[\phi]$.

A second notion of convergence is weak convergence of measures. From $\Lambda_n[\phi]$ we define a measure on $\mathbf{C}$

$$\mu_n[\phi] = \frac{1}{n} \sum_{\lambda\in\Lambda_n[\phi]} \delta_\lambda \tag{9}$$

with the understanding that the eigenvalues are counted with multiplicity. This measure is the sum of point masses concentrated at the eigenvalues, δ_λ being the Dirac delta concentrated at $\lambda \in \mathbf{C}$. (Note: considering $\Lambda_n[\phi]$ as counting the eigenvalues with multiplicity should not cause any confusion when thinking of it as a point in $\mathcal{H}(\mathbf{C})$.)

Now we define another measure in $\mathbf{C}$ that is supported on $\mathcal{R}[\phi]$, namely the push-forward measure $\phi_*(dx/(b-a))$ where dx is Lebesgue measure on the interval I. The measure of a subset $E \subset \mathbf{C}$ is the Lebesgue measure of $\phi^{-1}(E))$ divided by $b-a$. Equivalently, viewing $\phi_*(dx/(b-a))$ as a linear functional on the Banach space of bounded continuous functions on $\mathbf{C}$, we have

$$\phi_*(m)(f) = \frac{1}{b-a}\int_a^b f(\phi(x))dx \tag{10}$$

for any bounded continuous function $f : \mathbf{C} \to \mathbf{C}$. As the dual of a Banach space, the continuous linear functionals on the bounded continuous functions also form a Banach space. However the norm topology is too strong to allow the spectral convergence that we seek. Among the various weaker topologies we will use the weak* topology in which we define a sequence of measures μ_n to converge to μ, if $\lim_{n\to\infty} \mu_n(f) = \mu(f)$ for every bounded continuous f.

In the rest of this article we will summarize what is known about spectral convergence in both senses for the three examples. We will present numerical evidence for some conjectures and suggest what may be true, although it remains unproven. It helps to keep in mind that the holy grail in this subject is a pair of pseudo-theorems:

1. $\Lambda_n[\phi] \to \mathcal{R}[\phi]$ in $\mathcal{H}(\mathbf{C})$.
2. $\mu_n[\phi] \to \phi_*(m)$ weakly.

These are by no means true for all generating functions ϕ, but in some cases one or both of them is true. There is little that can be said in great generality for all multiplication operators and all orthonormal bases, but we do have the following.

Theorem 3.4. *For all n the eigenvalues of $M_n[\phi]$ are contained in the convex hull of $\mathcal{R}[\phi]$.*

Proof. Let λ be in the complement of the convex hull of $\mathcal{R}[\phi]$. Equivalently, 0 is not in the convex hull of $\mathcal{R}(\phi - \lambda)$. Pick a line L separating 0 from the convex hull of $\mathcal{R}(\phi - \lambda)$. Multiply by a non-zero $\gamma \in \mathbf{C}$ in order to rotate and compress the convex set so that the resulting set lies within the disk of radius 1 centered at 1. This set is bounded away from 0 since the line γL separates it from 0. Thus in the L^∞ norm (the uniform norm), $\|\gamma(\phi - \lambda)\| < 1$. It follows that in the operator norm $\|M[\gamma(\phi - \lambda) - 1]\| < 1$. Thus, $\|\gamma M[\phi - \lambda] - I\| < 1$. Now multiply the operator inside by the projection P_n on the left and restrict to the span of the first n basis elements. The projection P_n has norm less than 1, and so $\|\gamma M_n[\phi - \lambda] - I_n\| < 1$. A geometric series argument now shows that $\gamma M_n[\phi - \lambda]$ is invertible, and hence $M_n[\phi - \lambda]$ is, too. This, of course, is equal to $M_n[\phi] - \lambda I_n$, so λ is not an eigenvalue of $M_n[\phi]$. □

4. Spectral convergence for Toeplitz matrices

Example 4.1. Let $\phi(\theta) = e^{i\theta}$. Multiplication by $e^{i\theta}$ shifts the basis elements by one. The Toeplitz matrices are lower triangular with 1 on the sub-diagonal. All eigenvalues are zero; they spectacularly fail to converge to the unit circle in any way.

If ϕ has no negative (resp. positive) Fourier coefficients, then $T_n[\phi]$ is lower (resp. upper) triangular, and the spectrum consists of c_0 counted n times. In the first case ϕ is in H_∞ and in the second ϕ is in $\overline{H_\infty}$. In fact, if ϕ extends to an analytic function on an annulus containing the unit circle, then the spectrum of the finite matrices has little apparent relationship to the range of the function [10].

For positive results Szegö proved that if ϕ is real-valued, then $\mu_n[\phi] \to \phi_*(m)$ weakly [18]. Widom, whose work has been fundamental and wide-ranging in this subject for over 30 years, has proved that $\mu_n[\phi] \to \phi_*(m)$ weakly under the assumption that ϕ is continuous and piecewise C^∞ with one singularity so that it is not C^∞ [33].

For convergence of the eigenvalue sets in the space $\mathcal{H}(\mathbf{C})$ a theorem of Basor, following work of Böttcher and Silbermann, has established that for ϕ a smooth complex-valued function on $\mathbf{T}$ with one jump discontinuity, the sequence of sets of eigenvalues converge to the image of the function in $\mathcal{H}(\mathbf{C})$ [4]. When the results of Widom and Basor are combined we see that symbols that are smooth except for one discontinuity give Toeplitz matrices whose spectra converge both as sets in the Hausdorff metric and as measures. These results depend on the understanding of the asymptotic behavior of $\det(T_n[\phi] - \lambda)$ as $n \to \infty$. The conjecture of Fisher and Hartwig [14] arising in problems of statistical mechanics has been a motivation for much of the work in the last twenty-five years. Recently it has been shown that the conjecture is false with some simple counter examples, the simplest of which we consider below. A revised conjecture has been made by Basor and Tracy [5] and some progress has been made on it [3].

The case of symbols that extend to be analytic on an annulus containing the circle is quite interesting. Suppose $\phi(\theta) = \sum_k c_k e^{ik\theta}$ and that $\sum_k c_k z^k$ is convergent for $\rho < |z| < 1$. Then the function ϕ_r defined on the unit circle by $\phi_r(\theta) = \phi(r\theta)$ has the property that its Toeplitz matrices have the same eigenvalues as $T_n[\phi]$ because they are similar matrices using the diagonal matrix with entries $1, r, r^2, \ldots, r^{n-1}$. On the other hand, the images of ϕ (which is ϕ_1) and ϕ_r are not the same and there is no reason to prefer the image of any one of them over the others. The behavior for analytic functions does not appear to be typical. Widom has conjectured [33] that for a generic (in some suitable sense) set of symbols ϕ the eigenvalue measures converge weakly to $\phi_*(m)$. (This is what is meant by the eigenvalues being *canonically distributed.*)

In order to see Toeplitz matrices as finite approximations to multiplication operators we have to view the matrices as growing in all directions. That is, the finite matrices are the centers of the doubly infinite matrices representing the multiplication operator using the Fourier basis indexed by $n \in \mathbf{Z}$. If we look at the matrices as the upper left hand corners of an infinite matrix with rows and columns indexed by $n \in \mathbf{N}$, then the Toeplitz matrices are finite approximations to different operators, namely *Toeplitz operators.* To define a Toeplitz operator recall the the *Hardy space* $H_2(\mathbf{T})$ is the span of $e^{ik\theta}$, $k = 0, 1, 2, \ldots$. Let P be the projection of $L_2(\mathbf{T})$ onto $H_2(\mathbf{T})$. Then the Toeplitz operator $T[\phi] : H_2(\mathbf{T}) \to H_2(\mathbf{T})$ is defined to be $PM[\phi]|H_2(\mathbf{T})$. This puts us in the ambiguous position of having a family of matrices that naturally approximates two different operators, and these operators are, in fact, quite different. Their spectral pictures are different. The spectrum of the Toeplitz operator with symbol $e^{i\theta}$ is the closed unit disk, while the spectrum of the multiplication operator is the unit circle. (The Toeplitz operator is the unilateral shift, while the multiplication operator is the bilateral shift.) The spectrum of the Toeplitz operator with continuous symbol ϕ is the image of ϕ and its interior. (See [11, Ch. 7 Notes] for historical notes and references.) A generalization for piecewise continuous symbols holds [30]: connect the image of the function with straight lines between the left and right limits at the discontinuities, giving a closed curve, which together with its interior makes up the spectrum. In the eigenvalue pictures that follow one sees that the eigenvalues approach the image of the symbol but stay within the interior of the closed curve just described. There are **no** examples yet of matrix eigenvalues that clearly lie outside the spectrum of the associated Toeplitz operator. A deep theorem of Widom's [31], still not well-understood, shows that the spectrum of a Toeplitz operator is connected when the symbol is any L_∞ function. See [11] for an account of this theorem on the connectedness of the spectrum.

Suppose we want to approximate a Toeplitz operator by finite matrices. We have seen that the natural thing to do is to use the basis $e^{ik\theta}$, for $k \geq 0$. But then the matrices also approximate the multiplication operator. As the matrices grow in size their eigenvalues tend, typically, to the spectrum of the multiplication operator rather than to the spectrum of the Toeplitz operator. Is it, therefore, impossible to approximate a Toeplitz operator in this way? Would it be better to use a different basis? Could we use a different basis in a computationally effective way? What one would like to see is a sequence of matrices whose eigenvalues begin to fill up the interior in a uniform way. There are indications that, in some sense, only the essential spectrum of an operator is computable. See [1, 2] for some discussion of this. The *essential spectrum* of an operator A is the spectrum of $\overline{A}$ in the Banach algebra of bounded operators modulo the closed ideal of compact operators, which is known as the *Calkin algebra.* Now, the essential spectrum of $T[\phi]$ is the image of ϕ [11], and so the rest of the spectrum, which is the interior, may be inaccessible by numerical approximation. Clearly, these questions deserve further study.

Example 4.2. Let $\phi(\theta) = \theta(\cos\theta + i\sin\theta)$ whose image is a spiral. There is a single discontinuity at $\theta = 0$. Figure 2 shows both the image (solid curve) and the eigenvalues for $n = 51$. The work of Libby [19] proves that both the eigenvalue sets and the eigenvalue measures converge as they should according to the picture.

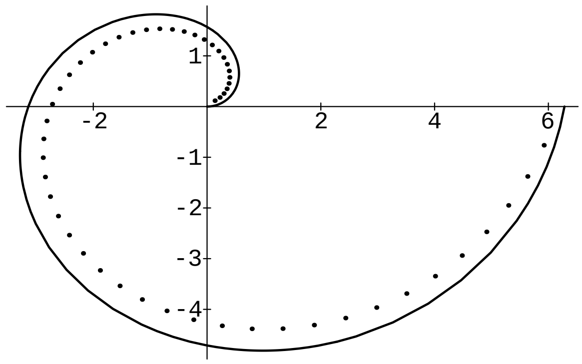

FIGURE 2. Eigenvalues of $T_{51}[\phi]$ and the image of ϕ of Example 4.2.

Example 4.3.

$$\phi(\theta) = \begin{cases} \pi/2 & \text{if } 0 < \theta < \pi \\ -\pi/2 & \text{if } -\pi < \theta < 0 \end{cases} \tag{11}$$

The Fourier coefficients are given by

$$c_n = \begin{cases} 0 & \text{if } n \text{ is even} \\ -i/n & \text{if } n \text{ is odd} \end{cases} \tag{12}$$

The Toeplitz matrix $T_n[\phi]$ is skew-symmetric and so for n odd there is always a zero eigenvalue. It is impossible for the eigenvalues to converge to the image! The multiplicity of the zero eigenvalue is one, however, and all the other eigenvalues cluster around $-\pi/2$ and $\pi/2$, half of them at each point. When n is even, half of the eigenvalues cluster at $-\pi/2$ and half at $\pi/2$. The eigenvalue measures approach the expected limit which is a measure with mass $1/2$ at $\pm\pi$.

Example 4.4. Define

$$\phi(\theta) = \begin{cases} \theta + i & \text{if } 0 < \theta < \pi \\ \theta + 2i & \text{if } \pi < \theta < 2\pi \end{cases} \tag{13}$$

This function has two discontinuities and the range is two disjoint line segments. Notice the difference between odd and even n and to the slight bending in the "curves" of eigenvalues where the real part is $1/2$. No theorem, as of yet, proves the convergence of the eigenvalue measures, but we believe that as n increases there will be more "stray" eigenvalues peeling away from the image, but not enough to affect the convergence of the eigenvalue measures to the measure supported on the image. The symmetry of the function enables one to show that for n odd there is always an eigenvalue of $\pi + 1.5i$ by translating the function by $-(\pi + 1.5i)$, which translates the eigenvalues by the same. Thus, the odd sized matrices are skew-symmetric. Of course, this means the sets of eigenvalues do not converge in $\mathcal{H}(\mathbf{C})$.

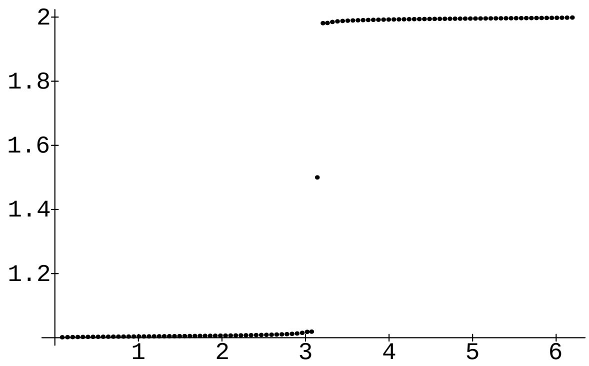

FIGURE 3. Eigenvalues of $T_{101}[\phi]$ of Example 4.4.

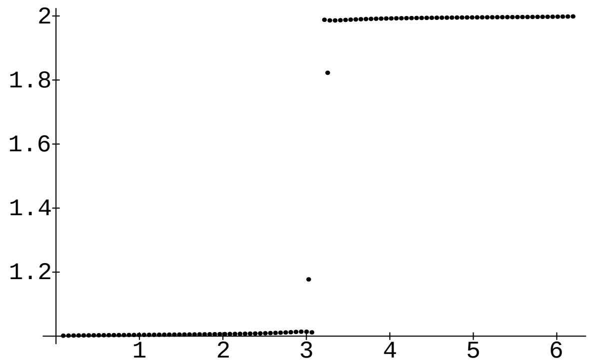

FIGURE 4. Eigenvalues of $T_{102}[\phi]$ of Example 4.4.

We end this section with a brief summary of *circulant matrices*, a special class of Toeplitz matrices. An $n \times n$ matrix is circulant if it has the form

$$\begin{bmatrix} c_0 & c_{n-1} & c_{n-2} & \cdots & c_1 \\ c_1 & c_0 & c_{n-1} & \cdots & c_2 \\ c_2 & c_1 & c_0 & \cdots & c_3 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ c_{n-1} & c_{n-2} & c_{n-3} & \cdots & c_0 \end{bmatrix} \tag{14}$$

Each row is a cyclic permutation of the row above and so an n by n circulant matrix depends on just n parameters. It has been known for a long time that the eigenvalues are exactly the values of the polynomial function $\sum_{k=0}^{n-1} c_k z^k$ evaluated at the nth roots of unity. This polynomial, thus, plays the role of the symbol with $z = e^{i\theta}$, but notice that the Toeplitz matrices associated to $\phi(\theta) = \sum_{k=0}^{n-1} c_k e^{ik\theta}$ are banded matrices with n non-zero diagonals and not circulant matrices. In [27] Toeplitz introduces what we now call "Toeplitz" matrices and uses the already understood circulant matrices to enrich his discussion of the eigenvalues of Toeplitz matrices.

It is tempting to attempt to use the explicit knowledge of the eigenvalues of circulant matrices to get information about the eigenvalues of general Toeplitz matrices by a process of approximation, but such an attempt is doomed as one can see with the example of $e^{i\theta}$. If we use the Euclidean norm to measure distance, then the circulant matrix nearest to $T_n[\phi]$ is

$$\begin{bmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & 0 \\ & & \vdots & & \\ 0 & 0 & \cdots & 1 & 0 \end{bmatrix} \tag{15}$$

Recall that $T_n[\phi]$ is the same matrix with the upper right hand one replaced by a zero. The eigenvalues of the circulant matrix are the nth roots of unity, while the eigenvalues of $T_n[\phi]$ are all zero.

Theorem 4.5. *Let C be a circulant matrix whose top row is $c_0, c_1, \ldots, c_{n-1}$. Then the eigenvalues of C are the values of $\sum_{k=0}^{n-1} c_k z^k$ as z ranges over the nth roots of unity $e^{2\pi i m/n}$, for $m = 0, 1, \ldots, n-1$.*

Proof. Although this theorem is ancient and well-known we include a proof based on the characters of the group $\mathbf{Z}_n$ because of the similarity with the Walsh-Toeplitz matrices. We consider $\mathbf{C}^n$ as the group algebra of $\mathbf{Z}_n$, i.e. the functions from $\mathbf{Z}_n$ to $\mathbf{C}$. The characters are those functions which are homomorphisms to the group of non-zero complex numbers. There are n characters χ_k, for $0 \le k \le n-1$, where $\chi_k(m) = e^{2\pi i k m/n}$, and they form an orthogonal basis of $\mathbf{C}^n$. (Here we are thinking of $\mathbf{Z}_n$ as being $\{0, 1, 2, \ldots, n-1\}$.) Let f be an element of the group algebra with $f(m) = d_m$. Consider the linear map "multiplication by f" on the group algebra. Using the standard basis of $\mathbf{C}^n$ the matrix of this linear map is the diagonal matrix with diagonal entries d_m. Now we leave it to the reader to check that with the basis of characters the matrix of this map is the circulant matrix whose jk coefficient is $c_{j-k} = \langle f, \chi_{j-k} \rangle$, where $j - k$ is interpreted mod n. Then the eigenvalues are $d_m = f(m) = \sum c_k \chi_k(m) = \sum c_k e^{2\pi i k m/n}$, which gives the theorem. □

5. Spectral Convergence with the Legendre Basis

We begin by considering the function $p_1(x) = \phi(x) = x$, which is the first Legendre polynomial and already normalized.

Theorem 5.1. *The eigenvalues of $M_n[p_1]$ are the n zeros of the Legendre polynomial p_n.*

Proof. We consider multiplication by x on the span of $p_0, p_1, \ldots, p_{n-1}$, but we use the basis of monomials $1, x, x^2, \ldots, x^{n-1}$, instead. With this basis multiplication by x shifts the basis elements to the right except for the last one. Now we multiply x^{n-1} by x and then project orthogonally. Let $p_n(x) = \sum_{k=0}^{n} \gamma_k x^k$. Then

$$x^n = \frac{1}{\gamma_n}\left(p_n(x) - \sum_{k=0}^{n-1} \gamma_k x^k\right)$$

from which it is clear that the projection onto the span of $p_0, p_1, \ldots, p_{n-1}$ is $(1/\gamma_n)(\sum_{k=0}^{n-1} \gamma_k x^k)$ and the matrix is the companion matrix for this polynomial. The eigenvalues of a companion matrix are the zeros of the polynomial and $M_n[p_1]$ is similar to this matrix. □

Szegö's classic work on orthogonal polynomials [26, Theorem 12.7.2] contains a theorem on the distribution of the zeros of orthogonal polynomials. For a fairly general weight function $w(x)$ on $[-1,1]$ the zeros of the nth orthogonal polynomial for this weight are distributed asymptotically like the values of $\cos(\theta)$ for equally spaced values of θ in $[0,\pi]$. Translated to the language of this article we see that the eigenvalue measures $\mu_n[p_1]$ converge weakly to the measure μ on $[-1,1]$ for which the measure of the interval $[a,b] \subset [-1,1]$ is $(\cos^{-1} b - \cos^{-1} a)/\pi$. The measure μ can be characterized as $dx/(\pi\sqrt{1-x^2})$, where dx denotes Lebesgue measure on the interval.

Some idea of the truth of this can be seen in the plots of the zeros of p_{30} and the values of $\cos(\theta)$ for thirty equally spaced values of θ between 0 and π.

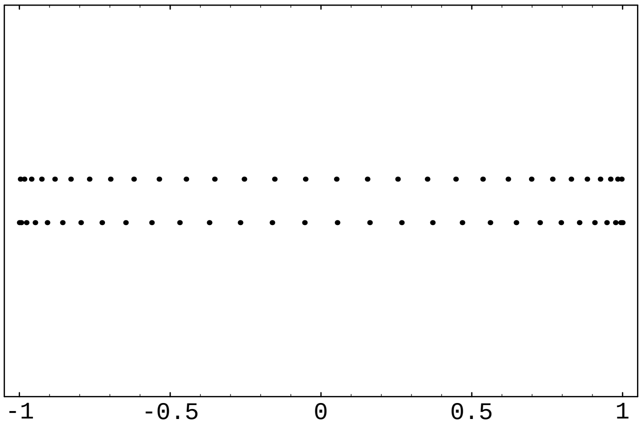

FIGURE 5. Eigenvalues of $M_{30}[p_1]$ and 30 values of $\cos(\theta)$ for equally spaced θ.

It is also true that the zeros of p_n converge to $[-1,1]$ in the space $\mathcal{H}(\mathbf{C})$, because the zeros of orthogonal polynomials lie in the interval of orthogonality and for every subinterval with endpoints a and b and n sufficiently large, p_n has a zero in $[a,b]$. Again, one should refer to [26].

The results for the multiplication operator associated to p_1 generalize to other real-valued functions ϕ. It is shown in [18] that the spectral measures converge.

Theorem 5.2. *Let ϕ be a real-valued L_∞ function on $[-1,1]$. Then the sequence of spectral measures $\mu_n[\phi]$ converges weakly to the measure μ defined by*

$$\mu([a,b]) = \frac{1}{\pi}(\phi(\cos^{-1} b) - \phi(\cos^{-1} a)).$$

Proof. See §6.2 in [18]. □

We have translated their statement to our language. Another way to state the conclusion is that for every continuous function f defined on $\mathbf{R}$ (it is sufficient that f be defined on the image of ϕ),

$$\lim_{n\to\infty} \frac{1}{n} \sum_{\lambda \in \Lambda_n[\phi]} f(\lambda) = \frac{1}{\pi}\int_{-1}^{1} \frac{f(\phi(x))}{\sqrt{1-x^2}}\, dx = \frac{1}{\pi}\int_0^{\pi} f(\phi(\cos\theta))\, d\theta \tag{16}$$

The limiting measure μ can also be characterized as $\phi_*(dx/\pi\sqrt{1-x^2})$.

The proof of this theorem and the related work of Szegö rely on the Hermitian nature of the matrices and the self-adjointness of the multiplication operators resulting from the symbols being real-valued. For complex valued ϕ it appears that almost nothing is known. It may be that the work of Widom, Basor, Böttcher, Silbermann, and others for Toeplitz matrices can be applied to the Legendre basis setting, but so far no one has done that. In the book of Grenander and Szegö [18] that is how the results for the Legendre basis are obtained as well as results for the orthogonal polynomials for other weights and the orthogonal polynomials on analytic curves, and hence it is suggestive for further work. A clear conjecture at which to aim, is that the previous theorem is true for any complex-valued function ϕ in L_∞. The danger in such a wide ranging conjecture is that for Toeplitz matrices the corresponding theorem holds for real-valued symbols and does not hold for complex-valued symbols as is shown by analytic ϕ such as $e^{i\theta}$. However, there are no clear counter-examples in the Legendre case. We consider that function in the next example.

Example 5.3. Define $\phi(x) = e^{i\pi x}$. Of course, the image is the unit circle. The plot shows the numerically approximated eigenvalues of $M_{30}[\phi]$.

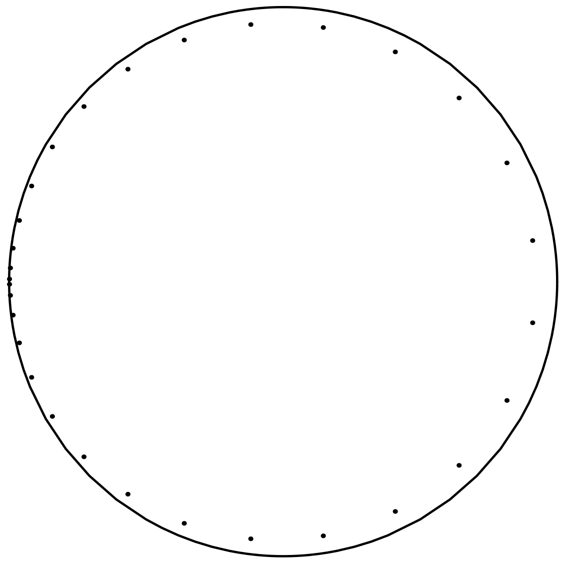

FIGURE 6. Eigenvalues of $M_{30}[\phi]$ and the unit circle of Example 5.3.

The plot shows that the eigenvalues are more concentrated in the region of the image for values of x near the ends of the interval. This gives some credibility to the conjecture that the distribution of the eigenvalues approaches the measure $\phi_*(dx/\pi\sqrt{1-x^2})$, but the conjecture has not been proved.

Example 5.4. Define $\phi(x) = \pi(x+1)e^{i\pi(x+1)}$ for $x \in [-1,1]$. The image is the spiral shown in Figure 2 and the function here is the modification of the function in that example to have the domain $[-1,1]$.

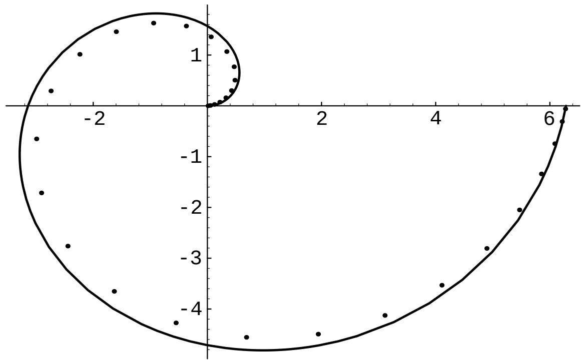

FIGURE 7. Eigenvalues of $M_{30}[\phi]$ and the image of ϕ of Example 5.4.

6. Spectral convergence for Walsh-Toeplitz matrices

Recall that we denote the Walsh functions by ψ_n, for $n \geq 0$.

Example 6.1. Let us consider the operator $M[\psi_1]$ and its associated finite matrices. Since $\psi_1\psi_k = \psi_{1\oplus k}$, where $\oplus$ denotes dyadic addition, we see that for k even $\psi_1\psi_k = \psi_{k+1}$, while for k odd $\psi_1\psi_k = \psi_{k-1}$. Thus $M_n[\psi_1]$ contains 2 by 2 diagonal blocks of the form

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{17}$$

when n is even, and when n is odd there is an extra row and column of zeros. Each 2 by 2 block contributes eigenvalues 1 and -1, so that for $n = 2p$ the eigenvalues of $M_n[\psi_1]$ are 1 and -1, each with multiplicity p, and for $n = 2p+1$ the eigenvalues are 1 and -1 each with multiplicity p and the value 0 with multiplicity 1. Thus the eigenvalue sets do not converge in the Hausdorff metric, but the eigenvalue distributions do converge to $(\delta_1 + \delta_{-1})/2$, which is the push-forward of Lebesgue measure on $[0,1]$ by ψ_1.

The previous example suggests that we can understand the eigenvalue behavior of $M_n[\psi_k]$ for any of the Walsh functions ψ_k.

Theorem 6.2. *The eigenvalue measures $\mu_n[\psi_k]$ converge weakly to $(\psi_k)_*(dx)$. The eigenvalue sets $\Lambda_n[\psi_k]$ do not converge to $\mathcal{R}[\phi] = \{-1, 1\}$ in $\mathcal{H}(\mathbf{C})$.*

Proof. Assume that k is between 2^{m-1} and 2^m. The Walsh functions ψ_j for $0 \leq j < 2^m$ form a group isomorphic to $\mathbf{Z}_2{}^m$. Thus, multiplication by ψ_k defines a permutation of order two on these 2^m basis elements. For each pair i and j for which $i \oplus k = j$ there is an eigenvector $\psi_i + \psi_j$ with eigenvalue 1 and an eigenvector $\psi_i - \psi_j$ with eigenvalue -1. When n is a multiple of 2^m, the matrix $M_n[\psi_k]$ consists of diagonal $2^m \times 2^m$ blocks that are repetitions of the first block. If n is not a multiple of 2^m, then there will be some values i such that $i \oplus k > n$, so that $M_n[\psi_k]$ will have a column of zeros in column i. Thus $M_n[\psi_k]$ will have an equal number of 1's and -1's in its spectrum and no more than 2^m zeros. As $n \to \infty$ the eigenvalue measure converges to $(\delta_1 + \delta_{-1})/2$, which is equal to $(\psi_k)_*(dx)$. $\square$

Now consider the finite sum $\phi(x) = \sum_{i=0}^{k} c_i\psi_i(x)$ with $k < 2^m$. The functions $\psi_0, \psi_1, \ldots, \psi_{2^m-1}$ are a basis of the space of functions on $[0,1]$ that are constant on the subintervals of length 2^{-m}. This space is invariant under multiplication by ϕ. To compute the eigenvalues of $M_{2^m}[\phi]$ let us use a different basis, namely the functions which take on the value 1 on one of the subintervals and 0 on the rest. These functions are eigenfunctions for multiplication by ϕ with eigenvalues equal to the values of ϕ on each of the subintervals. Now for n equal to a multiple of 2^m this pattern is repeated in each block of size 2^m. When n is not a multiple of 2^m, then there are some other eigenvalues, but no more than 2^m of them, and so as n goes to infinity their contribution to the eigenvalue measures goes to zero. The push-forward of Lebesgue measure by ϕ is the sum of point masses at the values of ϕ all divided by 2^m. Hence, we have proved the following theorem.

Theorem 6.3. *Let $\phi(x) = \sum_{i=0}^{k} c_i\psi_i(x)$ with k less than 2^m. Then*
(i) *$\mu_n[\phi]$ converges weakly to $\phi_*(dx)$.*
(ii) *For $n = 2^m$ and m sufficiently large, $\Lambda_n[\phi] = \mathcal{R}[\phi]$.*

Next we consider the question of extending these results to arbitrary continuous functions. We have much better understanding of the matrices $M_n[\phi]$ when n is a power of 2 because of the close relationship with the group ${\mathbf{Z}_2}^m$.

Theorem 6.4. *Let $P_n : L_2[0,1] \to \mathbf{H}_n$ where $\mathbf{H}_n$ is the span of ψ_k for $0 \le k \le n-1$ and P_n is the orthogonal projection onto this subspace. Furthermore, suppose $n = 2^m$. Then $P_n(f)$ is the locally constant function whose value on the interval $[2^{-m}(j-1), 2^{-m}j]$ is*

$$2^m \int_{2^{-m}(j-1)}^{2^{-m}j} f(x)\,dx$$

which is the average of f on this subinterval.

Proof. Let $\bar{f_n}$ denote this locally constant function defined by averaging. Then $\langle \bar{f_n}, \psi_k\rangle = \langle f, \psi_k\rangle$ for all k between 0 and $n-1$. □

For an arbitrary function ϕ in L_∞ we have nice eigenvalue behavior for the eigenvalue measures.

Theorem 6.5. *If ϕ is in $L_\infty[0,1]$, then the sequence of eigenvalue measures $\mu_{2^m}[\phi]$ converges weakly to $\phi_*(dx)$ as $m \to \infty$.*

Proof. Let f be any continuous function defined on $\mathbf{C}$. Let $\overline{\phi_{2^m}} = P_{2^m}(\phi)$, recalling that $\overline{\phi_{2^m}}$ is a step function whose value on each of the 2^m subintervals is the average of ϕ on the subinterval. Now $f \circ \overline{\phi_{2^m}}$ converges pointwise almost everywhere to $f\circ\phi$ and $\|f\circ\overline{\phi_{2^m}}\| \le \|f\circ\phi\|$. Therefore, by the Dominated Convergence Theorem we have

$$\lim_{m\to\infty} \int_0^1 f(\overline{\phi_{2^m}}(x))\,dx = \int_0^1 f(\phi((x))\,dx\ .$$

But the eigenvalues of $M_{2^m}[\phi]$ are the values of $\overline{\phi_{2^m}}$. Therefore,

$$\int_0^1 f(\overline{\phi_{2^m}}(x))\,dx = 2^{-m} \sum_{\lambda\in\Lambda_{2^m}[\phi]} f(\lambda).$$

□

Theorem 6.6. *If ϕ is continuous, then the sequence of eigenvalue sets $\Lambda_{2^m}[\phi]$ converges to the image of ϕ in $\mathcal{H}(\mathbf{C})$ as $m \to \infty$.*

Proof. The eigenvalues of $M_{2^m}[\phi]$ are the average values of ϕ on the subintervals of length 2^{-m}. Pick $\varepsilon > 0$. Since ϕ is actually uniformly continuous, for m sufficiently large, if x and y are in the same subinterval of length 2^{-m}, then $|\phi(x) - \phi(y)| < \varepsilon$. Hence, every value of ϕ is within ε of one of the average values and every average is within ε of a value of ϕ. □

What we have seen is that the eigenvalues behave as nicely as we could ever hope as long as we use the increasing family of subspaces of $L_2[0, 1]$ spanned by the first 2^m Walsh functions, as both the eigenvalue measures and the eigenvalue sets converge in their respective spaces to the "correct" limits for a continuous ϕ. For an arbitrary bounded measurable function the eigenvalue measures converge. The Walsh functions provide the ideal basis for numerical approximation of the spectrum of a large family of operators. But this suggests more tantalizing questions: does every bounded operator A have a well-adapted sequence of nested subspaces $\mathbf{H}_n$, whose union is dense in $\mathbf{H}$ so that the spectrum of A can be approximated by the eigenvalues of A_n? (We are not insisting that the dimension of $\mathbf{H}_n$ be n.) This leads us from the concrete setting of function spaces and multiplication operators to more general operator theory. For an introduction to this fascinating and little explored area see the recent work of Arveson dealing with self-adjoint operators [1, 2]. However, when n is not a power of 2, the eigenvalues are distributed differently as shown by the following pictures. It is not known whether the eigenvalue measures converge. The numerical evidence indicates that there are relatively few "stray" eigenvalues, so one may weakly conjecture that convergence does take place.

Example 6.7. Let $\phi(x) = e^{2\pi i x}$. The image of ϕ is the unit circle again. In each case the outer points lie barely within the unit circle, although they appear to be right on the circle.

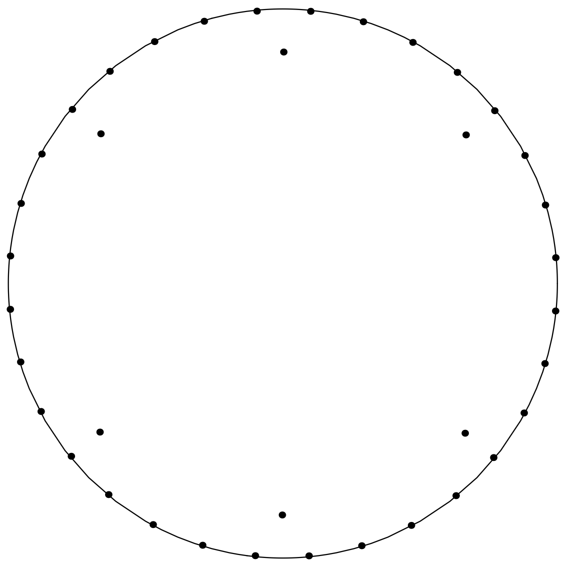

FIGURE 8. Eigenvalues of $M_{38}[\phi]$ in Example 6.7.

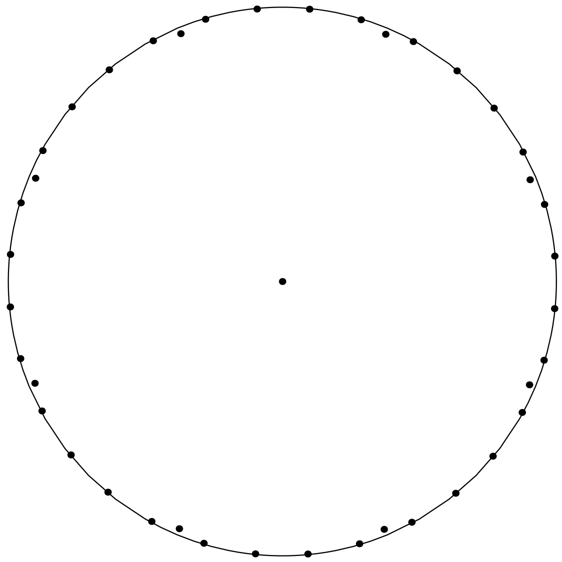

FIGURE 9. Eigenvalues of $M_{41}[\phi]$ in Example 6.7.

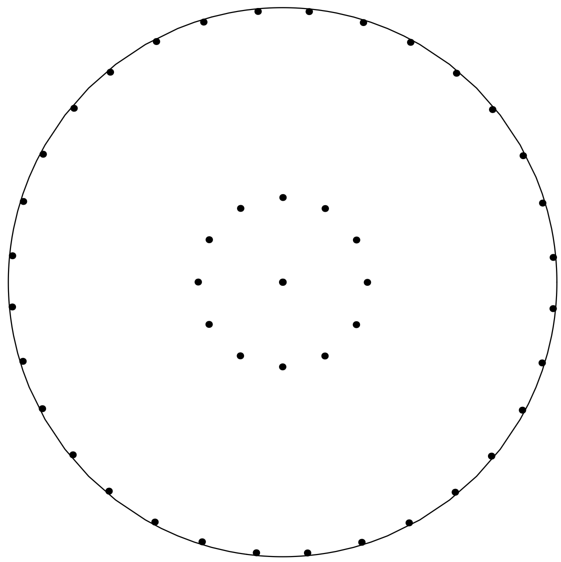

FIGURE 10. Eigenvalues of $M_{47}[\phi]$ in Example 6.7.

7. How the computations were done

The eigenvalues of the Toeplitz matrices were computed approximately using Mathematica running on Macintosh IIci or NeXT workstations. The Fourier coefficients of the generating functions were approximated by using the built-in fast Fourier transform in Mathematica and by sampling the function at 1024 equally spaced points from 0 to 2π. This allows hundreds of coefficients to be calculated quickly and quite accurately. A Toeplitz matrix of size 200 by 200 would require

401 of the Fourier coefficients, less than half of the computed coefficients. My worry in using this approach was that the approximate coefficients would be inaccurate, especially the higher order coefficients. But comparing the approximate coefficients done this way with exact coefficients computed by hand for some simple examples showed that the difference in the coefficients was less than 10^{-10}. The coefficients done by sampling were also compared with those computed by numerical integration (with the built-in routine) and found to differ by the same orders of magnitude. This, of course, is no guarantee of the accuracy of the numerical approximations, and everything done this way should be regarded as suggestive and provisional subject to proof by traditional means or to rigorous error estimates.

Eigenvalues are particularly sensitive to slight changes in the coefficients of the matrix. Interesting work along these lines deals with the *pseudo-eigenvalues* and *pseudo-spectrum* of matrices and operators [28, 24]. These results indicate that the pseudo-spectra of the finite Toeplitz matrices approximate the pseudo-spectrum of the Toeplitz operator.

As an example, consider $\phi(\theta) = e^{i\theta}$, whose associated Toeplitz matrices have a multiple eigenvalue of 0. When the Fourier coefficients are approximated by the discrete Fourier transform with 1024 sample points, we find $|c_k| < 10^{-18}$, for $k \neq 1$, but c_k is not exactly 0. One would expect the eigenvalues of the approximate $T_n[\phi]$ clustered closely around 0. However, for $n = 50$ the eigenvalues lie in a circle of radius 0.4. For smaller n the circle is smaller and for larger n the circle is larger. If $n = 1024$ then the matrix becomes a full circulant matrix and the eigenvalues are exactly the sample points.

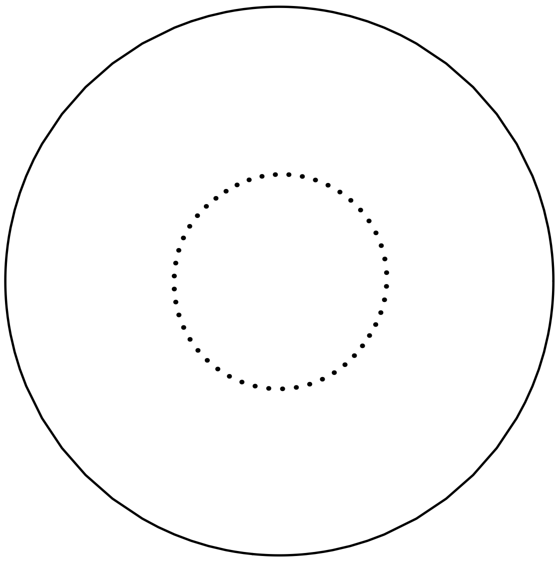

FIGURE 11. Spurious eigenvalues of $T_{50}[e^{i\theta}]$ and the unit circle.

There is no fast "Legendre transform" that approximates the Legendre basis coefficients of a function from sampled values. In order to compute the eigenvalues of $M_{30}[\phi]$ the numerical integration function of Mathematica was used along with the Legendre polynomials. This required for each function the evaluation of 900 integrals, but the symmetry reduced the number by half. The computation was done remotely on a Sun workstation and required several hours for each of the two examples. Then the matrix was saved to a file and transported to a Macintosh where the eigenvalue computation was done in Mathematica and the output plotted.

The Walsh examples were done with a combination of resources. I had earlier written a fast Walsh transform in True BASIC (to use in an engineering mathematics class) by modifying the fast Fourier transform algorithm in *Numerical Recipes* [23]. It takes a couple of seconds to perform a 1024 point real transform on a Quadra. So I used this for the real and imaginary parts separately and wrote the output to files. In Mathematica I read in the files and combined them to form the complex vector of coefficients. The rest was done in Mathematica and required a dyadic sum function and a function to construct the Walsh-Toeplitz matrix from the Walsh coefficients. The eigenvalues and plotting used the built-in functions.

It seemed prudent to cross-check the results obtained with Mathematica because the built-in functions, such as the eigenvalue routine, must be used as black boxes. One Toeplitz eigenvalue example, not used in this article but similar, was re-done using MATLAB and the numerical results were almost identical.

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