A New Discrete-Math Course based on the text "A Logical Approach to Discrete Math"

David Gries and Fred B. Schneider
Computer Science, Cornell University

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Our text "A Logical Approach to Discrete Math" attempts to change, in a revolutionary way, how discrete math is taught. Instead of simply teaching a collection of loosely related mathematical topics, a course taught from this text will give students a solid understanding of what constitutes a proof and a skill in developing, presenting, and reading proofs. To be sure, the traditional topics are still taught, but through the use of logic, rather than in place of or in addition to it. Logic is the glue that binds together proofs and arguments, in all domains. We make heavy use of this glue. (See Table 1 for a discussion of the displayed quotes.)

> *This course was really groovy, perhaps my favorite one .... I am a math major who never thought I would enjoy doing proofs, until taking this course.*
>
> *One of the best courses I've ever taken. It has taught me a lot about formal proof. It is essential for a CS major to learn about proof.*
>
> *Stressing logic has helped my methods of thinking. I definitely have a better understanding of proof.*
>
> *Before, my only experience with proofs and math induction was in high school. I did not really understand them; we just followed a formula to get the results wanted. This course really brought to my attention the real ideas behind proofs and math induction.*
>
> *If it was not required, I would have not taken this course, due largely to a distaste for proofs developed in high-school math courses. I viewed logic as pragmatically inapplicable to most real-world problems, since conditions in the real world seldom lend themselves to true-false definitions. But I now have a much greater appreciation for logic, both for its own sake and for its applications.*

Syntactic manipulation is a useful skill

One might view such an emphasis on proof as misguided, because it takes away precious time from other topics of the course —set theory, induction, relations and functions, combinatorics, graph theory, and the like. In the traditional discrete-math course (as popularized by most discrete-math texts), logic is just one more topic to be covered, and generally in isolation from the others. Naturally, students come away viewing logic as do the text and course, as another academic topic to wade through, with few applications. Further, the average student's mathematical ability is enhanced during the traditional course only by a surfeit of facts and not, as we would like, through heightened mathematical maturity and know-how.

We view the role of logic in a discrete-math course differently. We believe that teaching a skill in formal manipulation makes learning the other material easier. In fact, a skill in formal manipulation can be applied almost immediately in other scientific and engineering courses.

> *In my linear algebra class, I understand proofs with much less difficulty, and proving theorems about matrices has become easier as well.*
>
> *I've been using ... [the techniques] in other courses such as math.*
>
> *My math homework from last week ended with two proofs, and I was* happy *about it! My reasoning skills have become sharper.*
>
> *I have found myself making use of some of the things that I learned in this course to prove things in other courses (math and physics).*
>
> *My math professor writes a proof in English sentences. I usually do the same proof in my notes in a syntactically much simpler and clearer layout, without any long and wordy English sentences.*
>
> *The exposure to logic will help alleviate the workload in later math and CS courses.*

Equational logic is the best choice

Teaching a skill in proof and formal manipulation requires a logic. We believe that an equational logic, with substitution of equals for equals as the major inference rule, is the proper system to teach beginners. The equational style draws on familiarity with high-school algebra and (for those who have had it) trigonometry. Substitution of equals for equals is a kind of syntactic manipulation that computer scientists and mathematicians use all the time, and our text shows how it can be applied in many topics of discrete-math courses. The technical apparatus needed in order to use an equational logic is minimal and is easily learned. Moreover, the equational style can be used to impart useful heuristics and strategies for developing proofs, thus giving the students tools for thinking. Finally, the use of the equational style and the accompanying rigor can be largely free from overwhelming complexities and details —one of the disadvantages one often hears of other formal approaches.

People can easily use our equational logic, as opposed to some logics that are easier for computers to use. Our equational logic is a "People Logic", instead of a "Machine Logic".

> *I really enjoyed it; I've never seen such different topics and ideas discussed under one "roof": propositional and predicate calculus. It was interesting and, in some sense, amusing.*
>
> *There are real applications out there for predicate and propositional calculus and an observable benefit to formalism.*
>
> *Topics were often associated with some type of computer application, which is important for CS majors.*
>
> *One of the most interesting aspects of formalized logic is its applications to regular problems and word problems.*
>
> *I especially liked the applications to concrete examples.*

The displayed quotes in this chapter are from students who took a discrete-math course from one of the authors in Spring 1993. A draft of our text was used. Near the end of the course, the 70 students were asked to write a short essay explaining their views on what had been taught. Sixty-five of the students responded. About 90 percent of these essays were very positive about the course, although they did give constructive criticisms, as requested. For a perspective, we give below all the negative comments:

> *Course material: Yuk! I could see the purpose of the logic, but it all became too theory oriented. Now I know that I am not fond of logic, but I feel I have a good grasp of it.*
>
> *I was not particularly thrilled with the course. I did not like rigorously proving things I knew by common sense.*
>
> *Although formatted proofs make things very clear, I found myself tied up with details of the proof and confined to the specific format. I prefer writing proofs in words in which I think in paragraphs.*
>
> *The course material was too easy. I knew most of it before taking the course. The stress on logic was not helpful, because I was very familiar with formal logic before the class. I was also disappointed in the homework grading. Perfectly valid proofs were not accepted because they didn't follow the rigid form.*
>
> *I find this class tedious. I have always liked math and proofs, but this class was little more than drudgery. I just don't think the subject deserves an entire course of its own. The topics could easily be incorporated into other courses where they are relevant or useful. I learned notations for proofs, but I don't feel that I gained any new insights into proofs, proof techniques, or math in this class.*
>
> *This course is required, but I really had no interest in the material.*

Table 1: About the Quotes in this Chapter

Instill an appreciation for rigor

Many people agree that rigor is good as long as it does not get in the way, but they think that it always gets in the way! We think otherwise. In fact, rigor helps us in our own work, and we believe it can help students do mathematics. Our views on rigor echo those voiced by David Hilbert in his famous speech at the turn of the twentieth century: "It is an error to believe that rigor in proof is the enemy of simplicity. On the contrary, we find it confirmed by numerous examples that the rigorous method is at the same time the simpler and the more easily comprehended. The very effort for rigor forces us to discover simpler methods of proof. It also frequently leads the way to methods which are more capable of development than the old methods of less rigor."

For students to appreciate the value of rigor, they must be shown how rigor helps. This course, with its emphasis on equational logic, is a good place to do that. (On the other hand, today's typical introductory calculus course does not emphasize rigor and virtually ignores proof and understanding of the mathematics. Therefore, it does not fulfill the role that we believe introductory math courses should have.)

Teaching rigor means that rigor should be practiced by the instructor and required of the students. Teaching rigor calls for carefully laying out rules for syntactic manipulation, explaining their nuances; and then following these rules as a theory is constructed. Furthermore, it calls for formalizing all problems before trying to solve them. The cause is helped by showing definitions and rules from other texts that lack rigor and demonstrating the difficulties caused by this lack. Discussions of style, taste, and qualities like clarity, brevity, and simplicity —replete with many illustrations— can heighten students' awareness of the effective use of rigor. Gradually, students acquire new, considerably higher, standards for what constitutes a formal definition or a convincing proof.

> *I finally learned what proving something "rigorously" means, although it seems to me that it is not done very often.*
>
> *I really enjoyed ... [this course], because of the rigor and techniques of proof I learned.*
>
> *The course has helped me to think "logically and rigorously".*
>
> *I thought standardizing the syntax of proofs was a good feature. The emphasis on rigor and problem solving was another good feature, and I am definitely more mathematically mature now than 5 months ago.*

Instill confidence in syntactic manipulation

Critical to the success of our new approach to discrete math is for students to become comfortable manipulating symbols and formulas as uninterpreted objects. Initially, most students are troubled by the prospect of such uninterpreted manipulation. They want to think about the meanings of mathematical statements. Having meanings for objects is a safety net, which, students feel, prevents them from performing nonsensical manipulations. Put another way, students who prefer knowing what each object means probably (i) have no confidence in the soundness of the rules of manipulation or (ii) doubt their ability to perform syntactic manipulations prescribed by these rules. Unfortunately, the use of this particular safety-net does not scale well to complicated problems. Skill in performing uninterpreted syntactic manipulation does. (Formal manipulation also lends itself to support from mechanical aids, like computers, whereas interpreting formulas does not.) Thus, a primary goal of this course is to instill confidence in applying syntactic rules of manipulation.

> *After studying logic, I have a better understanding about the reasons behind a lot of math proofs, equations, formulas, theories. I found myself not only learning to solve a particular problem but also the reason for using a particular formula or theory to solve it, and also how to attack such problems in general.*

Logic requires an in-depth study

In covering any topic, an instructor must decide whether to paint it broadly or to take time with a fine brush. In the broad-brush approach, one moves quickly through the material, touching only on highlights and making interesting observations. This approach has two goals:

- To expose students to a subject and whet their appetite for further study.
- To acquaint students with the existence of methods and facts, so that they know where to turn when confronted with certain types of problems.

Depth of coverage requires going slowly and carefully through the material. The goals here are:

- To give the students a real understanding and skill in the subject.
- To build confidence in exercising that skill to solve problems.

Logic as a tool is so important to later work in computer science and mathematics that students must understand the use of logic and be sure in that understanding. Studying logic is a wonderful way for students to become comfortable with formalizing ideas and manipulating uninterpreted formulas. Therefore, our discrete-math course provides in-depth coverage of the propositional and predicate calculi.

Five to six weeks are spent on propositional and predicate calculus. The basic material is thoroughly covered, even though some students may have seen it earlier (but not so rigorously and not in this form). Frequent but shorter homeworks ensure that students get practice, for there is a big difference between reading a proof in a book or on the blackboard and writing one from scratch. Simple heuristics for developing syntactic proofs are studied. Interesting and different proofs from the students' homeworks are discussed in class. Some proofs are developed complete with false starts and dead ends, to show the students that designing a proof is not always a linear task, but may require backtracking.

Students learn that a logically correct answer may not suffice —answers must live up to aesthetic standards as well. Formulas and proofs are "write-once, read many" objects, it is worth the effort to explore alternative formulations. To this end, different proofs of a theorem may be compared for clarity, brevity, simplicity, and the like. One aim is to develop a proof that readers will find so obvious that they feel they could have developed it. Such a proof is "opportunity driven" or "forced", in that, at each step, the shapes of the formula, the goal, and the applicable theorems guide the development of that step in only one way. In such a proof, no rabbit is pulled out of a hat.

> *I have a better grasp now of some of the basic properties in mathematics that had been left out of my education.*
>
> *I had a lot of math background, and this course really helped clarify and enhance the ideas that I acquired in high school.*
>
> *I particularly enjoyed the way the course taught how to treat familiar math material in a more mathematically mature way (instead of just teaching new material).*
>
> *I developed a level of comfort with developing proofs that I had not previously achieved. The process of progressing through a problem logically one step at a time has enhanced my problem-solving skills.*
>
> *I certainly feel comfortable with proofs now. I can easily understand them when they are presented in a straightforward format, and I am also capable (to some extent) of writing them myself.*

Dispel the fear of mathematics

Many students, even those who aspire to be scientists and engineers, fear mathematics. They cannot easily see the relation between an English statement and its formalization in a mathematical notation, they can't read and understand a mathematical statement, and they don't understand what constitutes a proof. Mathematics is simply a mystery.

A by-product of the steady, thorough pace in this course is that students begin to lose their fear of mathematics. Partly, this is the result of taking time to explain notations and conventions that students may have seen but have never had explained. Partly, it is the result of using a more uniform and consistent notation. (For example, our single notation for all quantifications — summations, products, unions of families of sets, universal quantification, existential quantification, etc.— clarifies what up until now has been chaos.) And partly, it is the inevitable result of leading students slowly onward, giving them at each point the practice that they need to build confidence in their ability. Taken too fast or with the wrong view, logic can be frightening; taken at the right pace and with the right view, the use of logic as a tool is within the grasp of every student.

> *This course has been* extremely *beneficial to me. Before this semester, anything written in math syntax intimidated me. Now, I have a much better understanding of math notations, patterns, constructions. Proof methods I learned in this class have helped me a great deal in other courses.*
>
> *I am much more comfortable with math expressions and understanding proofs.*
>
> *The class dissolved my fear of proving things, and I find proofs easier to tackle now.*
>
> *The idea of going through propositional calculus slowly, making sure everyone understood, was very good, since skills learned there were used for the entirely of the class.*
>
> *I entered the course with doubt and came out with certainty.*
>
> *I found the course to be instructive and rewarding; as a result, I am looking forward . . . to applying for entrance to the CS major in the fall.*

Logic can be fun

The study of logic and discrete math can be made interesting and lively —it need not have the feeling of overburdening drudgery that is often evoked. However, the instructor must believe that logic and its application is fun and must try to instill enthusiasm for the subject in the students. A large part of this has to do not so much with the technical material but with the mood of the class. Do whatever is necessary to promote a relaxed atmosphere. Try to eliminate the feeling of competition. Tell the students that they are all capable of mastering the material (for they are). Joke. Relate interesting anecdotes about mathematicians and computer scientists (our text has 30-odd historical notes to help in this regard). At appropriate times, discuss current events in mathematics and computer science, to put today more into perspective.

> *. . . [this course] has been very stimulating and fun. It has provided a much stronger foundation for my math background.*
>
> *To be perfectly honest, I enjoyed it, which is surprising considering I do not enjoy math.*
>
> *I thoroughly enjoyed, and found extremely useful, the material in in the course.*
>
> *When I told people I was going to take this course, more than a few said "good luck". In retrospect, I can't see what problems they had.*
>
> *Many people who had taken the course previously said it was a lot of grunt work and not very interesting. I'm pleased to say that I didn't find this too be the case at all.*
>
> *I thought the course would be too hard or boring or beyond me. But, after having taken the course, I can say that it was not that bad; in fact, it was pretty good.*
>
> *. . . [the instructor] took a course notorious for dull, dry material and gave it a little bit of life.*
>
> *The way the proof methods for propositional calculus fit into predicate calculus was quite interesting.*
>
> *I had never been subject to formal logic before, and I found the approach refreshing. My favorite topic was propositional calculus.*

The general effect of learning a useful formalism

One benefit of our new approach to teaching logic, we believe, is that students become more effective in communicating and thinking in other scientific and engineering disciplines. Rigor, precision, thoroughness, clarity, and brevity are not just owned by formalists, but should be qualities towards which all strive. Nevertheless, one can best get a good handle on these qualities through experience in working with an effective formalism.

> *Stressing logic is gravely important. There are too many people going around saying simple concepts in complex, unwieldy ways.*
>
> *Overall, I felt this course has helped my analytical, logical thinking.*
>
> *... propositional calculus has trickled into my English so that I am more careful when ... [I speak.]*
>
> *This course taught me the most about how —to think logically— in all the many courses I have taken in my Cornell career.*

Success with the new approach

The payoff in using the new approach to discrete math can be substantial. Students actually lose their fear of math, and attribute it to the course! Students now feel they understand proof and have confidence in their ability to develop and understand proofs! They use their new-found skill in other courses before the discrete-math course ends!

At least, that is what the students in our course felt, as the displayed quotes in this chapter indicate (see Table 1).

We would like to see the same kind of leap made in all discrete-math courses. Armed with the text "A Logical Approach to Discrete Math", you should be able to duplicate or surpass our success.