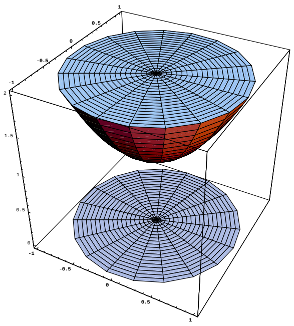
8-25-1993

Review: Double Integrals.

Example. Set up double integrals for the volume of the region $x^2 + y^2 + 1 \le z \le 2$ by: (a) Projecting the region into the x-y plane; (b) Projecting the region into the y-z plane. Use one of the integrals to compute the volume.

Here is the region:



The "bowl" is the surface $z = x^2 + y^2 + 1$.

The projection of the region into the x-y plane is the interior of the circle $x^2 + y^2 = 1$. I'll convert to polar. The projection is

$$0 \le r \le 1, \quad 0 \le \theta \le 2\pi.$$

To find the volume, I integrate top − bottom, which is

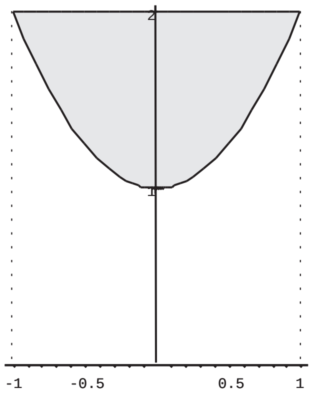
$$2 - (x^2 + y^2 + 1) = 1 - x^2 - y^2 = 1 - r^2.$$

Since I'm converting to polar, I replace $dx\,dy$ with $r\,dr\,d\theta$. The volume is

$$V = \int_0^{2\pi} \int_0^1 (1 - r^2) r\,dr\,d\theta = \int_0^{2\pi} \int_0^1 (r - r^3)\,dr\,d\theta = \int_0^{2\pi} \left[\frac{1}{2}r^2 - \frac{1}{4}r^4\right]_0^1 d\theta =$$

$$\frac{1}{4}\int_0^{2\pi} d\theta = \frac{\pi}{2}.$$

Here is the projection of the region into the y-z-plane:



The y-z-plane is $x = 0$, so the curve in the picture is $z = y^2 + 1$. So the following inequalities describe the region:

$$-1 \le y \le 1, \quad y^2 + 1 \le z \le 2.$$

What is the integrand? Turn the picture of the solid (above) on its side so that x is up. If I solve $z = x^2 + y^2 + 1$ for x, I get $x = \pm\sqrt{z^2 - y^2 - 1}$. The "top" is the equation with the plus sign, while the "bottom" is the equation with the minus sign. The integrand is

$$\text{top} - \text{bottom} = \sqrt{z^2 - y^2 - 1} - (-\sqrt{z^2 - y^2 - 1}) = 2\sqrt{z^2 - y^2 - 1}.$$

The integral for the volume is

$$V = \int_{-1}^{1} \int_{y^2+1}^{2} 2\sqrt{z^2 - y^2 - 1}\, dz\, dy. \quad \square$$

A thin sheet of metal (or something) occupies the region R in the plane. The material's density at the point (x, y) is $\delta(x, y)$. The mass of the sheet is

$$M = \iint_R \delta(x, y)\, dx\, dy.$$

The **center of mass** $(\overline{x}, \overline{y})$ is given by

$$\overline{x} = \frac{\int\int_R x \cdot \delta(x, y)\, dx\, dy}{M}$$

$$\overline{y} = \frac{\int\int_R y \cdot \delta(x, y)\, dx\, dy}{M}$$

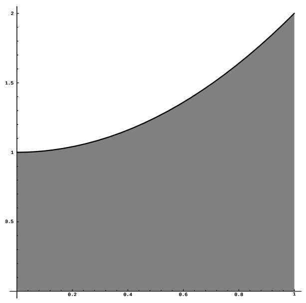
(The numerators of the fractions on the right sides are the **x-moment** and **y-moment**.)

Example. A sheet of metal occupies the region in the first quadrant below the curve $y = x^2 + 1$, from $x = 0$ to $x = 1$. Find the coordinates of its center of mass if its density is proportional to the square of the distance from the origin.

The square of the distance to the origin is $x^2 + y^2$, so the density is $\delta = k(x^2 + y^2)$, where k is a constant. The region is

$$0 \le x \le 1, \quad 0 \le y \le x^2 + 1.$$

Picture:



The mass is

$$\int_0^1 \int_0^{x^2+1} k(x^2 + y^2)\, dy\, dx = \frac{152k}{105}.$$

The x-moment is

$$\int_0^1 \int_0^{x^2+1} kx(x^2 + y^2)\, dy\, dx = \frac{25k}{24}.$$

The y-moment is

$$\int_0^1 \int_0^{x^2+1} ky(x^2+y^2)\,dy\,dx = \frac{94k}{63}.$$

The coordinates of the center of mass are

$$\overline{x} = \frac{875}{1216} \approx 0.71957, \quad \overline{y} = \frac{235}{228} \approx 1.03070. \square$$