Math 228
3-20-1994

Review Problems on Laplace transforms.

Example. Compute $\mathcal{L}\left[(t+1)u_2(t)\right]$.

Match the expression against the step function transform formula:

$$(t+1)u_2(t) = f(t-c)u_c(t).$$

This means $c = 2$, so $f(t-2) = t+1$. I need $f(t)$; to get it, just replace "t" with "$t+2$" in $f(t-2) = t+1$:

$$f[(t+2)-2] = (t+2)+1, \quad f(t) = t+3.$$

Therefore,

$$F(s) = \mathcal{L}\left[f(t)\right] = \frac{1}{s^2} + \frac{3}{s}.$$

Finally, by the step function shift,

$$\mathcal{L}\left[(t+1)u_2(t)\right] = \left(\frac{1}{s^2} + \frac{3}{s}\right) e^{-2s}. \ \Box$$

Example. Compute $\mathcal{L}\left[t^2\delta(t-1)\right]$.

The impulse function transform formula says

$$\mathcal{L}\left[g(t)\delta(t-c)\right] = g(c)e^{-cs}.$$

In this case, $c = 1$ and $g(t) = t^2$. So

$$\mathcal{L}\left[t^2\delta(t-1)\right] = 1^2 \cdot e^{-s} = e^{-s}. \ \Box$$

Example. Compute $\mathcal{L}^{-1}\left(\dfrac{s}{s^2+4s+5}\right)$.

Complete the square:

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+4s+5}\right) = \mathcal{L}^{-1}\left(\frac{s}{(s+2)^2+1}\right).$$

Since I have an $s+2$ on the bottom, I will try to write everything in terms of $s+2$:

$$\mathcal{L}^{-1}\left(\frac{s}{(s+2)^2+1}\right) = \mathcal{L}^{-1}\left(\frac{(s+2)-2}{(s+2)^2+1}\right) = \mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) - \mathcal{L}^{-1}\left(\frac{2}{(s+2)^2+1}\right).$$

For the first term,

$$\frac{s+2}{(s+2)^2+1} \quad \text{is} \quad F(s+2) \quad \text{if} \quad F(s) = \frac{s}{s^2+1}.$$

In this case,

$$f(t) = \mathcal{L}^{-1}\left[F(s)\right] = \cos t.$$

Hence, by the exponential shift,

$$\mathcal{L}^{-1}\left(\frac{s+2}{(s+2)^2+1}\right) = e^{-2t}\cos t.$$

For the second term,

$$\frac{2}{(s+2)^2+1} \quad \text{is} \quad G(s+2) \quad \text{if} \quad G(s) = \frac{2}{s^2+1}.$$

Now

$$g(t) = \mathcal{L}^{-1}[G(s)] = 2\sin t.$$

Hence, by the exponential shift,

$$\mathcal{L}^{-1}\left(\frac{2}{(s+2)^2+1}\right) = 2e^{-2t}\sin t.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{s}{s^2+4s+5}\right) = e^{-2t}\cos t - 2e^{-2t}\sin t. \quad \square$$

Example. Compute $\mathcal{L}^{-1}\left(\dfrac{(s+1)e^{-2s}}{s^2+2s+2}\right)$.

The e^{-2s} says I have a step function shift with $c = 2$. I need to compute $\mathcal{L}^{-1}\left(\dfrac{s+1}{s^2+2s+2}\right)$. Complete the square:

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2+2s+2}\right) = \mathcal{L}^{-1}\left(\frac{s+1}{(s+1)^2+1}\right).$$

$$\frac{s+1}{(s+1)^2+1} \quad \text{is} \quad G(s+1) \quad \text{if} \quad G(s) = \frac{s}{s^2+1}.$$

Now

$$g(t) = \mathcal{L}^{-1}[G(s)] = \cos t.$$

By the exponential shift,

$$\mathcal{L}^{-1}\left(\frac{s+1}{s^2+2s+2}\right) = e^{-t}\cos t.$$

Finally, I apply the step function shift. Replace t with $t-2$ and tack on the step function:

$$\mathcal{L}^{-1}\left(\frac{(s+1)e^{-2s}}{s^2+2s+2}\right) = e^{-(t-2)}\cos(t-2)u_2(t). \quad \square$$

Example. Solve $y'' + y = h(t)$, $y(0) = 0$, $y'(0) = 0$, if

$$h(t) = \begin{cases} 4 & \text{if } 0 \le t < 2 \\ t+2 & \text{if } t \ge 2 \end{cases}$$

First, I'll write $h(t)$ in terms of step functions:

$$h(t) = 4(u_0 - u_2) + (t+2)u_2 = 4u_0 + (t-2)u_2.$$

The transform of h is

$$\mathcal{L}\left(4u_0 + (t-2)u_2\right) = \frac{4}{s} + \frac{1}{s^2}e^{-2s}.$$

The first term is immediate. For the second term, match the expression against the step function transform formula:

$$(t-2)u_2(t) = f(t-c)u_c(t).$$

This means $c = 2$, so $f(t-2) = t-2$. Therefore, $f(t) = t$, and $F(s) = \mathcal{L}\left[f(t)\right] = \dfrac{1}{s^2}$. Hence,

$$\mathcal{L}\left[(t-2)u_2(t)\right] = \frac{1}{s^2}e^{-2s}.$$

Now that I have the transform of h, I'll go back and transform both sides of the differential equation. I apply the initial value formula on the left, and the result I derived for h on the right:

$$\left(-0 - 0\cdot s + s^2Y(s)\right) + Y(s) = \frac{4}{s} + \frac{1}{s^2}e^{-2s}.$$

Therefore,

$$Y(s) = \frac{4}{s(s^2+1)} + \frac{1}{s^2(s^2+1)}e^{-2s}.$$

The first term could be inverted by doing partial fractions. I think the convolution formula is a little easier. Note that

$$\mathcal{L}^{-1}\left(\frac{4}{s}\right) = 4 \quad \text{and} \quad \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t.$$

So by the convolution formula,

$$\mathcal{L}^{-1}\left(\frac{4}{s(s^2+1)}\right) = 4 \star \sin t = \int_0^t 4\sin u\,du = \left[-4\cos u\right]_0^t = 4(1-\cos t).$$

(I plugged $t-u$ into 4 — which gives 4 — and u into $\sin t$. Notice how easy it is to have a constant function there!)

To invert the second term, first note that e^{-2s} means I have a step function shift with $c = 2$. I have to invert $\dfrac{1}{s^2(s^2+1)}$. On approach is to use convolution. However, the simple form allows me to get the partial fractions decomposition by inspection:

$$\frac{1}{s^2(s^2+1)} = \frac{1}{s^2} - \frac{1}{s^2+1}.$$

Therefore,

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}\right) = t - \sin t.$$

Apply the step function shift. Replace t with $t-2$ and tack on u_2:

$$\mathcal{L}^{-1}\left(\frac{1}{s^2(s^2+1)}e^{-2s}\right) = \left[(t-2) - \sin(t-2)\right]u_2(t).$$

Therefore, the solution is

$$y(t) = 4(1-\cos t) + \left[(t-2) - \sin(t-2)\right]u_2(t).\ \square$$

Example. Use Laplace transforms to compute

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2\,ds.$$

First,

$$\int_0^\infty \left(\frac{\sin s}{s}\right)^2\,ds = \int_0^\infty (\sin s)^2\cdot\frac{1}{s^2}\,ds = \frac{1}{2}\int_0^\infty (1-\cos 2s)\cdot\frac{1}{s^2}\,ds.$$

Now $\mathcal{L}(t) = \dfrac{1}{s^2}$, so the integral becomes:

$$\frac{1}{2}\int_0^\infty (1-\cos 2s)\cdot\mathcal{L}(t)\,ds = \frac{1}{2}\int_0^\infty (1-\cos 2s)\cdot\left(\int_0^\infty te^{-st}\,dt\right)\,ds = \frac{1}{2}\int_0^\infty t\left(\int_0^\infty (1-\cos 2s)e^{-2t}\,ds\right)\,dt.$$

Uisng integration by parts,

$$\int_0^\infty (1-\cos 2s)e^{-2t}\,ds = \left[-\frac{1}{t}e^{-st} + \frac{4}{4+t^2}e^{-2t}\left(\frac{1}{2}\sin 2s - \frac{1}{4}t\cos 2s\right)\right]_0^\infty = \frac{4}{t(t^2+4)}.$$

So

$$\frac{1}{2}\int_0^\infty t\left(\int_0^\infty (1-\cos 2s)e^{-2t}\,ds\right)\,dt = \frac{1}{2}\int_0^\infty t\cdot\frac{4}{t(t^2+4)}\,dt = \frac{1}{2}\int_0^\infty \frac{4}{4+t^2}\,dt =$$

$$\frac{1}{2}\left[2\tan^{-1}\frac{t}{2}\right]_0^\infty = \frac{\pi}{2}.\ \square$$