Math 228
2-16-1994

Solutions to Problem Set 12.

[page 150]

19. Use the method of reduction of order to find a second solution of the differential equation $x^2y'' + 2xy' - 2y = 0$, $x > 0$, if $y_1 = x$ is a particular solution.

Let $y = vy_1 = vx$, so $y' = v'x + v$ and $y'' = v''x + 2v'$.
Substitute:

$$\begin{array}{rcllll} x^2y'' + 2xy' - 2y & = & x^3v'' & + & 2x^2v' & \\ & & & + & 2x^2v' & + \; 2xv \\ & & & & & - \; 2xv \\ & = & x^3v'' & + & 4x^2v'. & \end{array}$$

Thus,

$$x^3v'' + 4x^2v' = 0.$$

Simplify:

$$xv'' + 4v' = 0, \quad v'' + \frac{4}{x}v' = 0.$$

Substitute $u = v'$, $u' = v''$ to reduce the order:

$$u' + \frac{4}{x}u = 0.$$

Separate variables:

$$\int \frac{du}{u} = \int -\frac{4}{x}\,dx, \quad \ln|u| = -4\ln|x| + C, \quad u = C_0x^{-4}.$$

Set $u = v'$, then integrate to find v:

$$v' = C_0x^{-4}, \quad v = -\frac{1}{3}C_0x^{-3} + C_2.$$

Put y back:

$$y = vx = -\frac{C_0}{3}x^{-2} + C_2x = C_1 \cdot \frac{1}{x^2} + C_2x. \; \square$$

---

21. Use the method of reduction of order to find a second solution of the differential equation $x^2y'' - x(x+2)y' + (x+2)y = 0$, $x > 0$, if $y_1 = x$ is a particular solution.

Let $y = vy_1 = vx$, so

$$y' = v'x + v \quad \text{and} \quad y'' = v''x + 2v'.$$

Substitute:

$$\begin{array}{rcllll} x^2y'' - x(x+2)y' + (x+2)y & = & x^3v'' & + & 2x^2v' & \\ & & & - & x^2(x+2)v' & - \; x(x+2)v \\ & & & & & + \; x(x+2)v \\ & = & x^3v'' & - & x^3v'. & \end{array}$$

Thus,

$$v'' - v' = 0.$$

This is a linear constant coefficient homogeneous equation with characteristic equation $m^2 - m = 0$. The roots are $m = 0$ and $m = 1$, so the solution is

$$v = c_1 + c_2 e^x.$$

Therefore,

$$y = vx = c_1 x + c_2 x e^x. \square$$

---

22. Use the method of reduction of order to find a second solution of the differential equation $xy'' - y' + 4x^3 y = 0$, if $y_1 = \sin(x^2)$ is a particular solution.

Let $y = vy_1 = v\sin(x^2)$, so $y' = v'\sin(x^2) + 2xv\cos(x^2)$ and $y'' = v''\sin(x^2) + 4xv'\cos(x^2) + 2v\cos(x^2) - 4x^2 v\sin(x^2)$.

Substitute:

$$\begin{array}{rclcccc} xy'' - y' + 4x^3 y & = & xv''\sin(x^2) & + & 4x^2 v'\cos(x^2) & & \\ & & & + & 2xv\cos(x^2) & - & 4x^3 v\sin(x^2) \\ & & & - & v'\sin(x^2) & - & 2xv\cos(x^2) \\ & & & & & + & 4x^3 v\sin(x^2) \\ & = & xv''\sin(x^2) & + & \left(4x^2\cos(x^2) - \sin(x^2)\right) v'. & & \end{array}$$

Thus,

$$xv''\sin(x^2) + \left(4x^2\cos(x^2) - \sin(x^2)\right) v' = 0.$$

Let $u = v'$, so $u' = v''$:

$$xu'\sin(x^2) + \left(4x^2\cos(x^2) - \sin(x^2)\right) u = 0.$$

Separate variables:

$$\int \frac{du}{u} = \int \left(-4x\frac{\cos(x^2)}{\sin(x^2)} + \frac{1}{x}\right) dx, \quad \ln|u| = -2\ln|\sin(x^2)| + \ln|x| + C = \ln\left|\frac{x}{(\sin(x^2))^2}\right| + C.$$

Exponentiate both sides:

$$u = \frac{C_0 x}{(\sin(x^2))^2}.$$

Put v back and integrate:

$$v' = \frac{C_0 x}{(\sin(x^2))^2} = C_0 x(\csc(x^2))^2, \quad v = -\frac{C_0}{2}\cot(x^2) + C_2.$$

Put y back:

$$y = v\sin(x^2) = -\frac{C_0}{2}\cos(x^2) + C_2\sin(x^2) = C_1\cos(x^2) + C_2\sin(x^2). \square$$

---

25. Use the method of reduction of order to find a second solution of the differential equation $x^2 y'' + xy' + (x^2 - 0.25)y = 0$, $x > 0$, if $y_1 = x^{-1/2}\sin x$ is a particular solution.

Let $y = vx^{-1/2}\sin x$. Then

$$y' = v' x^{-1/2}\sin x + v\left(-\frac{1}{2}x^{-3/2}\sin x + x^{-1/2}\cos x\right)$$

$$y'' = v''x^{-1/2}\sin x + v'\left(-x^{-3/2}\sin x + 2x^{-1/2}\cos x\right) + v\left(\frac{3}{4}x^{-5/2}\sin x - x^{-3/2}\cos x - x^{-1/2}\sin x\right)$$

Plug these into the original equation to reduce the order and solve:

$$x^2y'' + xy' + \left(x^2 - \frac{1}{4}\right)y =$$

$$v''x^{3/2}\sin x + v'\left(-x^{1/2}\sin x + 2x^{3/2}\cos x\right) + v\left(\frac{3}{4}x^{-1/2}\sin x - x^{1/2}\cos x - x^{3/2}\sin x\right) +$$

$$v'x^{1/2}\sin x + v\left(-\frac{1}{2}x^{-1/2}\sin x + x^{1/2}\cos x\right) + v\left(x^{3/2}\cos x - \frac{1}{4}x^{-1/2}\sin x\right) =$$

$$v''x^{3/2}\sin x + 2v'x^{3/2}\cos x = 0$$

Since $x > 0$, it is legal to cancel $x^{3/2}$. Then

$$v''\sin x + 2v'\cos x = 0.$$

Let $u = v'$, so $u' = v''$:

$$u'\sin x + 2u\cos x = 0.$$

Separate variables:

$$\int \frac{du}{u} = -2\int \frac{\cos x}{\sin x}\,dx, \quad \ln|u| = -2\ln|\sin x| + C, \quad u = C_0(\csc x)^2.$$

Put v back and integrate:

$$v' = C_0(\csc x)^2, \quad v = C_1\cot x + C_2.$$

Put y back:

$$y = vx^{-1/2}\sin x = C_1x^{-1/2}\cos x + C_2x^{-1/2}\sin x. \ \square$$

---

26. The differential equation

$$xy'' - (x+N)y' + Ny = 0,$$

where N is a nonnegative integer, has been discussed by several authors. One reason it is interesting is that it has an exponential solution and a polynomial solution.

(a) Verify that one solution is $y_1 = e^x$.

$$\begin{aligned} y_1 &= e^x, \quad Ny_1 = Ne^x \\ y_1' &= e^x, \quad -(x+N)y_1' = -xe^x - Ne^x \\ y_1'' &= e^x, \quad xy_1'' = xe^x \end{aligned}$$

Therefore,

$$xy_1'' - (x+N)y_1' + Ny_1 = xe^x - xe^x - Ne^x + Ne^x = 0. \ \square$$

(b) Show that a second solution has the form $y_2(x) = ce^x \int x^N e^{-x}\,dx$. Calculate $y_2(x)$ for $N = 1$ and $N = 2$; convince yourself that, with $c = -1/N!$,

$$y_2(x) = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \cdots + \frac{x^N}{N!}.$$

Note that $y_2(x)$ is exactly the first $N+1$ terms in the Taylor series about $x = 0$ for e^x, that is, for $y_1(x)$. Let $y = vy_1 = ve^x$, so $y' = v'e^x + 6ve^x$ and $y'' = v''e^x + 2v'e^x + ve^x$. Substitute:

$$\begin{aligned} xy'' - (x+N)y' + Ny &= xe^x v'' + \begin{array}{ccc} 2xe^x v' & + & xe^x v \\ -(x+N)v'e^x & - & (x+N)ve^x \\ & + & Nve^x \end{array} \\ &= xe^x v'' + (x-N)e^x v'. \end{aligned}$$

Thus,

$$xe^x v'' + (x-N)e^x v' = 0.$$

Simplify to obtain $xv'' + (x-N)v' = 0$. Now let $u = v'$, so $u' = v''$:

$$xu' + (x-N)u = 0.$$

Separate variables:

$$\frac{du}{dx} = \left(\frac{N}{x} - 1\right) u, \quad \int \frac{du}{u} = \int \left(\frac{N}{x} - 1\right) dx, \quad \ln|u| = N \ln|x| - x + C = \ln|x|^N - x + C, \quad u = C_0 x^N e^{-x}.$$

Put v back and integrate:

$$v' = C_0 x^N e^{-x}, \quad v = C_0 \int x^N e^{-x}\, dx.$$

Put y back:

$$y = ve^x = e^x C_0 \int x^N e^{-x}\, dx.$$

Suppose $N = 1$. Then

$$y = e^x C_0 \int xe^{-x}\, dx = C_0 e^x \left(-xe^{-x} - e^{-x} + C_1\right) = C_2(x+1) + C_3 e^x.$$

Suppose $n = 2$. Then

$$y = e^x C_0 \int x^2 e^{-x}\, dx = C_0 e^x \left(-x^2 e^{-x} - 2xe^{-x} - 2e^{-x} + C_1\right) = C_2 \left(\frac{1}{2}x^2 + x + 1\right) + C_3 e^x. \ \square$$

"There is no such thing on earth as an uninteresting subject; the only thing that can exist is an uninterested person." - G. K. Chesterton

