Math 228
2-19-1994

Solutions to Problem Set 13.

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3. Find the general solution of the differential equation $y'' - 2y' - 3y = -3xe^{-x}$.

The homogeneous equation is $(D^2 - 2D - 3)y_h = 0$, or $(D-3)(D+1)y_h = 0$. The homogeneous solution is

$$y_h = c_1 e^{3x} + c_2 e^{-x}.$$

Next, find a particular solution. The right side xe^{-x} is a solution to $(D+1)^2 y = 0$. Apply $(D+1)^2$ to both sides:

$$(D+1)^2(D-3)(D+1)y_p = (D+1)^2(-3xe^{-x}) = 0, \quad y_p = Ae^{-x} + Bxe^{-x} + Cx^2e^{-x} + De^{3x}.$$

Ae^{-x} and De^{3x} are homogeneous solutions; throw them out. Compute the derivatives:

$$y_p = Bxe^{-x} + Cx^2e^{-x},$$

$$y_p' = Be^{-x} - Bxe^{-x} + 2Cxe^{-x} - Cx^2e^{-x} = Be^{-x} + (-B+2C)xe^{-x} - Cx^2e^{-x},$$

$$y_p'' = -2Be^{-x} + Bxe^{-x} + 2Ce^{-x} - 2Cxe^{-x} - 2Cxe^{-x} + Cx^2e^{-x} = (-2B+2C)e^{-x} + (B-4C)xe^{-x} + Cx^2e^{-x}.$$

Substitute:

$$\begin{array}{rcccccc} y_p'' - 2y_p' - 3y_p & = & (-2B+2C)e^{-x} & + & (B-4C)xe^{-x} & + & Cx^2e^{-x} \\ & & -2Be^{-x} & - & 2(-B+2C)xe^{-x} & + & 2Cx^2e^{-x} \\ & & & - & 3Bxe^{-x} & - & 3Cx^2e^{-x} \end{array}$$

$$= (-4B+2C)e^{-x} - 8Cxe^{-x} = -3xe^{-x}.$$

Therefore, $-4B + 2C = 0$ and $-8C = -3$. These equations yield $B = \dfrac{3}{16}$, $C = \dfrac{3}{8}$. The particular solution is

$$y_p = \frac{3}{16}xe^{-x} + \frac{3}{8}x^2e^{-x}.$$

The general solution is

$$y = c_1 e^{3x} + c_2 e^{-x} + \frac{3}{16}xe^{-x} + \frac{3}{8}x^2e^{-x}. \square$$

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4. Find the general solution of the differential equation $y'' + 2y' = 3 + 4\sin 2x$.

The homogeneous equation is $D(D+2)y_h = 0$. The homogeneous solution is

$$y_h = c_1 + c_2 e^{-2x}.$$

The particular solution y_p should satisfy

$$D(D+2)y_p = 3 + 4\sin 2x.$$

3 is killed by D and $\sin 2x$ is killed by $D^2 + 4$. Apply $D(D^2+4)$ to both sides:

$$D^2(D^2+4)(D+2)y_p = 0, \quad y_p = A + Bx + C\cos 2x + D\sin 2x + Ee^{-2x}.$$

A and Ee^{-2x} are homogeneous solutions — throw them out. The derivatives are

$$y_p = Bx + C\cos 2x + D\sin 2x,$$

$$Dy_p = B - 2C\sin 2x + 2D\cos 2x,$$

$$D^2y_p = -4C\cos 2x - 4D\sin 2x.$$

Plug these into the original equation:

$$(D^2 + 2D)y_p = 2B + (-4C + 4D)\cos 2x + (-4D - 4C)\sin 2x = 3 + 4\sin 2x.$$

$2B = 3$ so $B = \dfrac{3}{2}$. $-4C + 4D = 0$ gives $C - D = 0$ and $-4C - 4D = 4$ gives $C + D = -1$. These equations yield $C = -\dfrac{1}{2}$ and $D = -\dfrac{1}{2}$. The particular solution is

$$y_p = \frac{3}{2}x - \frac{1}{2}\cos 2x - \frac{1}{2}\sin 2x.$$

The general solution is

$$y = c_1 + c_2e^{-2x} + \frac{3}{2}x - \frac{1}{2}\cos 2x - \frac{1}{2}\sin 2x. \square$$

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13. Find the solution of the initial value problem $y'' + y' - 2y = 2x$, $y(0) = 0$, $y'(0) = 1$.

The homogeneous equation is $(D + 2)(D - 1)y_h = 0$. The homogeneous solution is

$$y_h = c_1e^{-2x} + c_2e^{x}.$$

The particular solution y_p satisfies

$$(D + 2)(D - 1)y_p = 2x.$$

Since $2x$ is killed by D^2, apply D^2 to both sides:

$$D^2(D + 2)(D - 1)y_p = 0, \quad y_p = A + Bx + Ce^{-2x} + De^{x}.$$

Ce^{-2x} and De^{x} are homogeneous solutions — throw them out. The derivatives are

$$y_p = A + Bx, \quad Dy_p = B, \quad D^2y_p = 0.$$

Plug these into the original equation:

$$(D^2 + D - 2)y_p = 0 + B - 2(A + Bx) = (B - 2A) + 2Bx = 2x.$$

Therefore, $B - 2A = 0$ and $-2B = 2$. Hence, $B = -1$, $A = -\dfrac{1}{2}$. The particular solution is

$$y_p = -\frac{1}{2} - x.$$

The general solution is

$$y = c_1e^{-2x} + c_2e^{x} - \frac{1}{2} - x.$$

Plug in the initial conditions:

$$0 = y(0) = c_1 + c_2 - \frac{1}{2}.$$

Since $y' = -2c_1 e^{-2x} + c_2 e^x - 1$,

$$1 = y'(0) = -2c_1 + c_2 - 1.$$

It follows that $c_1 = -\frac{1}{2}$ and $c_2 = 1$. The solution is

$$y = -\frac{1}{2}e^{-2x} + e^x - \frac{1}{2} - x. \square$$

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20. Determine a suitable form for $Y(x)$ for the equation $y'' + y = x(1 + \sin x)$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

A particular solution satisfies

$$(D^2 + 1)y_p = x(1 + \sin x) = x + x\sin x.$$

x is killed by D^2; $\sin x$ is killed by $D^2 + 1$, but since the term is $x \sin x$, the root is doubled. Therefore, apply $(D^2 + 1)^2$ to both sides:

$$D^2(D^2 + 1)^2(D^2 + 1)y_p = D^2(D^2 + 1)^2(x + x\sin x) = 0,$$

$$y_p = A + Bx + C\cos x + D\sin x + Ex\cos x + Fx\sin x + Gx^2\cos x + Hx^2\sin x.$$

Since $C\cos x$ and $D\sin x$ are homogeneous solutions, throw them out. Try

$$y_p = A + Bx + Ex\cos x + Fx\sin x + Gx^2\cos x + Hx^2\sin x. \square$$

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21. Determine a suitable form for $Y(x)$ for the equation $y'' - 5y' + 6y = e^x\cos 2x + e^{2x}(3x + 4)\sin x$ if the method of undetermined coefficients is to be used. Do not evaluate the constants.

A particular solution satisfies

$$(D^2 - 5D + 6)y_p = e^x\cos 2x + e^{2x}(3x + 4)\sin x.$$

$e^x\cos 2x$ corresponds to a root $1 + 2i$, and is killed by a factor $D^2 - 2D + 5$. $e^{2x}\sin x$ corresponds to a root $2 + i$, and is killed by $D^2 - 4D + 5$. However, since the term is $e^{2x}(3x + 4)\sin x$, the root is doubled. Therefore, apply $(D^2 - 4D + 5)^2$ to both sides:

$$(D^2 - 4D + 5)^2(D^2 - 2D + 5)(D^2 - 5D + 6)y_p = 0,$$

$$y_p = e^{2x}(A\cos x + B\sin x) + e^{2x}(Cx\cos x + Dx\sin x) + e^x(E\cos 2x + F\sin 2x) + Ge^{3x} + He^{2x}.$$

Since e^{3x} and e^{2x} are homogeneous solutions, throw them out. Try

$$y_p = e^{2x}(A\cos x + B\sin x) + e^{2x}(Cx\cos x + Dx\sin x) + e^x(E\cos 2x + F\sin 2x). \square$$

"I have always found that the less we speak of our intentions, the more chance there is of realizing them." - JOHN RUSKIN

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