Math 228
2-28-1994

Addendum: Solutions to Problem Set 16.

[page 279]

This problem was inadvertently omitted from the solutions.

26. (d) Show that for $p > 0$

$$p(p+1)(p+2)\cdots(p+n-1) = \frac{\Gamma(p+n)}{\Gamma(p)}.$$

Thus $\Gamma(p)$ can be determined for all positive values of p if $\Gamma(p)$ is known in a single interval of unit length, say $0 < p \le 1$. It is possible to show that $\Gamma\left(\dfrac{1}{2}\right) = \sqrt{\pi}$. Find $\Gamma\left(\dfrac{3}{2}\right)$ and $\Gamma\left(\dfrac{11}{2}\right)$.

For $n = 1$, the formula says

$$\frac{\Gamma(p+1)}{\Gamma(p)} = p.$$

This is true, by part (a).

Now suppose $n > 1$, and suppose the formula holds for $n - 1$. This means that

$$\frac{\Gamma(p+n-1)}{\Gamma(p)} = p(p+1)(p+2)\cdots(p+n-2).$$

Then

$$\frac{(p+n-1)\cdot\Gamma(p+n-1)}{\Gamma(p)} = p(p+1)(p+2)\cdots(p+n-2)(p+n-1).$$

By part (a), this is

$$\frac{\Gamma(p+n)}{\Gamma(p)} = p(p+1)(p+2)\cdots(p+n-2)(p+n-1).$$

Thus, the formula is true for all $n > 0$, by induction.

Taking $n = 1$ and $p = \dfrac{1}{2}$,

$$\Gamma\left(\frac{3}{2}\right) = \Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2} = \frac{\sqrt{\pi}}{2}.$$

Likewise, taking $n = 5$ and $p = \dfrac{1}{2}$,

$$\Gamma\left(\frac{11}{2}\right) = \Gamma\left(\frac{1}{2}\right)\cdot\frac{1}{2}\cdot\frac{3}{2}\cdot\frac{5}{2}\cdot\frac{7}{2}\cdot\frac{9}{2} = \frac{945\sqrt{\pi}}{32}. \square$$